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WADD TECHNICAL REPORT 60-133

CYLINDRICAL SANDWICH CONSTRUCTION DESIGN

Sidney Allinikov

Materials Laboratory

FEBRUARY 1960

WRIGHT AIR DEVELOPMENT DIVISION

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FOREWORD

This report was prepared by the Design Criteria Branch, Materials Laboratrry, Directorate of Laboratories, Wright Air Development Division. Preparation of the report was initiated under Project No. 7381, "Materials Application," Task No. 73812, "Data Collection and Correlation," with Sidney Allinikov acting as project engineer.

The compilation of papers presented herein represent a significant effort in the development of design data applicable to cylindrical sandwich constructions. The purpose of this report is to provide wide distribution of these particular papers, which has not been done previously.

ABSTRACT

This report is a compilation of papers which present a comprehensive treatment of the theories and parameters associated with the design of cylindrical sandwich constructions. Many of the formulas developed are applicable to a wide variety of core and facing combinations. Experimental data on flat and curved sandwich sections are furnished to support the theoretical solutions related to the design of these structures.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

D. A. SHINN
Chief, Design Criteria Branch
Applications Division
Materials Laboratory

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SECTION I

ELASTIC STABILITY OF CYLINDRICAL SANDWICH SHELLS UNDER AXIAL AND LATERAL LOAD[1]

By

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Summary

A linear solution for the determination of the loads under which a cylindrical sandwich shell will buckle is presented. The facings of the sandwich cylinder are treated as cylindrical shells and the core as an orthotropic elastic body. The method of solution is of interest in that it is of sufficient generality to be applied to many problems in sandwich analysis. The characteristic determinant that represents the solution to the problem is solved numerically. Curves that show how the buckling load changes as the parameters of the problem change are given.

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[1] This report is one of a series prepared by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics Order No. 01593.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

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Introduction

Sandwich construction is a result of the search for a strong, stiff, and yet light weight material. It is usually made by gluing relatively thin sheets of a strong material to the faces of relatively thick but light weight, and often weak, material. The outer sheets are called "facings" and the inner layer is called the "core."

Such a layered system presents difficult design problems. What is offered here is a straightforward method for dealing with some of these problems.

The problem to which the method is applied is that of the elastic stability of a sandwich cylinder under uniform external lateral load and uniform axial load.

Notation

r, θ, z	radial, tangential, and longitudinal coordinates, respectively
a	radius to middle surface of outer facing
b	radius to middle surface of inner facing
t	thickness of each facing
ℓ	length of cylinder
E	modulus of elasticity of facings
μ	Poisson's ratio of facings
G	modulus of rigidity of facings
E_c	modulus of elasticity of core in direction normal to facings
$G_{r\theta}$	modulus of rigidity of core in $r\theta$ plane
G_{rz}	modulus of rigidity of core in rz plane
q	intensity of uniform external lateral loading
k	$\dfrac{1}{1 + \dfrac{b}{a} - \dfrac{Et \log \frac{b}{a}}{E_c a}}$
σ_r	normal stress in core in radial direction
$\tau_{r\theta}$, τ_{rz}	transverse shear stresses in core
u, v, w	radial, tangential, and longitudinal displacements, respectively
n	number of waves in circumference of buckled cylinder
m	number of half waves in length of buckled cylinder

λ	$\dfrac{m\pi a}{\ell}$
$\delta_{n\theta}$	$\dfrac{E_c}{2G_{r\theta}} - \dfrac{n^2}{2}$
δ_z	$\dfrac{E_c}{G_{rz}}$
N_θ, N_z, $N_{\theta z}$	normal forces and shear force per unit length of facing
Q_θ, Q_z	transverse shear forces per unit length of facing
M_θ, M_z	bending moments per unit length of facing
$M_{z\theta}$, $M_{\theta z}$	twisting moments per unit length of facing
R, θ, Z	surface forces per unit area of facing
β	$\dfrac{E_c a\,(1 - \mu^2)}{Et}$
ϕ_1	$\dfrac{qa\,(1 - \mu^2)}{Et}$
ϕ_2	$\dfrac{N_z\,(1 - \mu^2)}{Et}$
α	$\dfrac{t^2}{12a^2}$
α'	$\dfrac{t^2}{12b^2}$
log	natural logarithm

A, B, C, D, K, L, A', B', A'', B'' arbitrary constants

note -- any of the above terms that appear with a prime (as N_z') refer to the inner facing.

Mathematical Analysis

As previously stated, the core is relatively weak. Because of the high strength of the facings the core need carry little tension or compression except in a direction perpendicular to the facings. The facings are able to resist shearing deformation in their plane and it is necessary only that the core be able to resist shear in the radial direction in planes perpendicular to the facings. In this analysis the core is considered to be an orthotropic elastic body. It is unable to resist deformations other than those just mentioned. This assumption makes it possible to determine explicitly how the stresses vary throughout the thickness of the core.

The facings are treated as shells.

Interdependance of the core and the facings is gained by equating their displacements at the interfaces. To simplify the analysis the core is assumed to extend to the middle surface of each facing.

Figure 1 shows the cylinder and the coordinates that are used.

Prebuckling Stresses

Before buckling occurs the cylinder is in a state of uniform compression. The axial load is carried by the facings since the core material is assumed to be incapable of carrying load in this direction. With facings of like material the stress is the same in both facings.

If, in addition, the facings have the same thickness, then the loading per unit length of facing, N_z or N_z', will be the same. This means that for a total load **P**,

$$2\pi a\, N_z + 2\pi b\, N_z' = P.$$

The calculation of stresses due to the lateral pressure is a problem in rotational symmetry. Differential elements of the core and of the facings are shown in figure 2.

Summing forces in the radial direction gives for the core

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} = 0,$$

for the outer facing

$$aq - a\,(\sigma_r)_{r=a} - N_\theta = 0,$$

and for the inner facing

$$b\,(\sigma_r)_{r=b} - N_\theta' = 0.$$

Since $\sigma_r = E_c \dfrac{\partial u}{\partial r}$,

$$N_\theta = Et\left(+\frac{u}{a}\right)_{r=a}, \text{ and}$$

$N_\theta' = Et\left(+\dfrac{u}{b}\right)_{r=b}$, these equations can be solved for σ_r, N_θ and N_θ'. The results are*

$$\sigma_r = q\,\frac{a}{r}\,k$$

$$N_\theta = qa\,(1 - k), \text{ and}$$

*For a more detailed derivation of these terms see Reference 1.

$N_\theta' = qak$ where

$$k = \frac{1}{1 + \frac{b}{a} - \frac{Et \log \frac{b}{a}}{E_c a}}$$

As P and q increase, N_z, N_z', N_θ, N_θ', and σ_r also increase. Eventually a condition may be reached where a slight increase in load causes the cylinder to lose its state of uniform compression and buckle as a result of elastic instability. This buckling is assumed to cause only a small change in the stress distribution. These small changes will now be considered.

Buckling Stresses

The Core

A free body diagram of an element of the core is shown in figure 3.

Neglecting terms which are products of more than three differentials, a summation of forces in the r, θ, and z direction gives

$$\sigma_r + r\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} + r\frac{\partial \tau_{rz}}{\partial z} = 0 \tag{1}$$

$$r\frac{\partial \tau_{ra}}{\partial r} + 2\tau_{r\theta} = 0 \tag{2}$$

$$\tau_{rz} + r\frac{\partial \tau_{rz}}{\partial r} = 0 \tag{3}$$

Equation (2) may be integrated to give

$$\tau_{r\theta} = \frac{B}{r^2} f_1 (\theta) f_1 (z) \tag{4}$$

Equation (3) may be integrated to give

$$\tau_{rz} = \frac{A}{r} f_2 (\theta) f_2 (z) \tag{5}$$

σ_r, $\tau_{r\theta}$ and τ_{rz} as defined in terms of u, v, and w are

$$\sigma_r = E_c \frac{\partial u}{\partial r} \tag{6}$$

$$\tau_{r\theta} = G_{r\theta} \left[\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right] \tag{7}$$

$$\tau_{rz} = G_{rz} \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right] \tag{8}$$

It is convenient to assume the displacements u, v and w in the form

$$u = f_1 (r) \cos n\theta \cos \frac{\lambda}{a} z \tag{9}$$

$$v = f_2 (r) \sin n\theta \cos \frac{\lambda}{a} z \tag{10}$$

$$w = f_3 (r) \cos n\theta \sin \frac{\lambda}{a} z \tag{11}$$

This form will permit a unique determination of f_1 (r), f_2 (r), and f_3 (r); assumes upon buckling n circumferential waves and m longitudinal half waves; results in zero displacements in the radial and circumferential directions at the ends; and imposes no moment upon the facings at the ends.

From a consideration of equations (4), (5), (7), (8), (9), (10) and (11) it is clear that

$$f_1(\theta)\, f_1(z) = \sin n\theta \cos \frac{\lambda}{a} z \qquad \text{and}$$

$$f_2(\theta)\, f_2(z) = \cos n\theta \sin \frac{\lambda}{a} z, \qquad \text{so that}$$

$$\tau_{r\theta} = \frac{B}{r^2} \sin n\theta \cos \frac{\lambda}{a} z \quad \text{and} \tag{12}$$

$$\tau_{rz} = \frac{A}{r} \cos n\theta \sin \frac{\lambda}{a} z. \tag{13}$$

Substituting equation (9) into (6) and then equations (6), (12) and (13) into equation (1) gives

$$E_c \frac{\partial f_1(r)}{\partial r} + E_c r \frac{\partial^2 f_1(r)}{\partial r^2} + \frac{nB}{r^2} + \frac{\lambda}{a} A = 0 \tag{14}$$

which upon integration shows that

$$f_1(r) = C + D \log r + A' r + B' \frac{1}{r} \tag{15}$$

Equations (9), (10) and (12) are substituted into equation (7) to give

$$\frac{B}{r^2} = G_{r\theta} \left[\frac{n}{r}\left(C + D \log r + A' r + \frac{B'}{r}\right) + \frac{\partial f_2(r)}{\partial r} - \frac{f_2(r)}{r} \right], \tag{16}$$

from which

$$f_2(r) = Fr + Cn + Dn(1 + \log r) + A' n\, r \log r + \frac{B'' n}{r}. \tag{17}$$

Equations (9), (11) and (13) are substituted into equation (8) to give

$$\frac{A}{r} = G_{rz} \left[C + D \log r + A' r + B' \frac{1}{r} + \frac{\partial f_3(r)}{\partial r} \right], \tag{18}$$

from which

$$f_3(r) = K + A'' (r^2 + \log r) + Cr + Dr(\log r - 1)$$

$$+ B \log r. \tag{19}$$

It is convenient to have the constants of $f_1(r)$, $f_2(r)$ and $f_3(r)$ in non-dimensional form. Redefining the constants the following form is obtained.

$$u = \left(Aa + Br + C\frac{a^2}{r} + Da \log\frac{r}{a}\right) \cos n_\theta \cos\frac{\lambda}{a} z \tag{20}$$

$$v = \left[-Ana + Bnr \log\frac{r}{a} + C\frac{a^2}{nr}\,\delta_{n\theta} - Dan\left(\log\frac{r}{a} + 1\right) + Fr \right] \sin n\theta \cos\frac{\lambda}{a} z \tag{21}$$

$$w = \left[A\lambda r + Ba\lambda\left(\frac{r^2}{2a^2} - \frac{\delta_z}{\lambda^2} \log\frac{r}{a}\right) + C\lambda a \log\frac{r}{a} + D\lambda r \left(\log\frac{r}{a} - 1\right) + La \right] \cos n\theta \sin\frac{\lambda}{a} z \tag{22}$$

The Facings

Free body diagrams of a facing element showing the forces and moments are shown in figures 4 and 5. It is necessary, in this type of problem, to include components of forces which result from elastic deformation of the element. The geometry of the situation is such that it is difficult to write equations of equilibrium. It is safest to use results obtained from a mathematical theory of thin shells. Such theory, as developed by Osgood and Joseph (ref. 2), when applied to cylindrical shells yields, for the outer facing at r = a, the following equations

$$\Sigma F_z = 0 = a \frac{\partial N_z}{\partial z} + \frac{\partial N_{\theta z}}{\partial \theta} + N_{z\theta} \frac{\partial^2 w}{\partial z \partial \theta} - N_\theta \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) - a\, Q_z \frac{\partial^2 u}{\partial z^2} - Q_\theta \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right) + aZ \tag{23}$$

$$\Sigma F_\theta = 0 = a \frac{\partial N_{z\theta}}{\partial z} + \frac{\partial N_\theta}{\partial \theta} - Q_z \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right) + Q_\theta \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} \right) - N_z \frac{\partial^2 w}{\partial z \partial \theta}$$

$$+ N_{\theta z} \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) + a\, \theta \tag{24}$$

$$\Sigma F_r = 0 = a \frac{\partial Q_z}{\partial z} + \frac{\partial Q_\theta}{\partial \theta} + a\, N_z \frac{\partial^2 u}{\partial z^2} + N_{\theta z} \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right) + N_{z\theta} \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z} \right)$$

$$- N_\theta \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} \right) + a\, R \tag{25}$$

$$\Sigma M_z = 0 = a \frac{\partial M_{z\theta}}{\partial z} - \frac{\partial M_\theta}{\partial \theta} + M_z \frac{\partial^2 w}{\partial z \partial \theta} - M_{\theta z} \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) - a\, Q_\theta - a\, Q_z \frac{\partial v}{\partial z} \tag{26}$$

$$\Sigma M_\theta = 0 = a \frac{\partial M_z}{\partial z} + \frac{\partial M_{\theta z}}{\partial \theta} - M_{z\theta} \frac{\partial^2 w}{\partial z \partial \theta} - M_\theta \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) + a\, Q_z + Q_\theta \frac{\partial w}{\partial \theta} \tag{27}$$

$$\Sigma M_r = 0 = M_z \left(\frac{\partial^2 u}{\partial z \partial \theta} - \frac{\partial v}{\partial z}\right) - M_\theta \left(\frac{\partial^2 u}{\partial \theta \partial z} - \frac{\partial v}{\partial z}\right) + a M_{z\theta} \frac{\partial^2 u}{\partial z^2} - M_{\theta z} \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{a \theta}\right) -$$

$$a(N_{z\theta} - N_{\theta z}) - a(N_z - N_\theta)\left(\frac{\partial v}{\partial z} + \frac{1}{a} \frac{\partial w}{\partial \theta}\right) \qquad (28)$$

As is customary in such problems the stretching of the middle surface is taken into account by substituting in equations (23) to (28)

$N_z\ (1 + \epsilon_\theta)$ for N_z,

$N_\theta\ (1 + \epsilon_z)$ for N_θ,

and multiplying the surface forces by

$(1 + \epsilon_\theta)\ (1 + \epsilon_z)$.

In these expressions

$$\epsilon_\theta = \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{u}{a} \quad \text{and}$$

$$\epsilon_z = \frac{\partial w}{\partial z}.$$

N_z and N_θ of equations (23) to (28) are replaced by $\left(\frac{P}{2\pi\ (a + b)} + \Delta N_z\right)$ and $[qa\ (1 - k) + \Delta N_\theta]$, and in the corresponding equations for the inner facing N_z' and N_θ' are replaced by $\left(\frac{P}{2\pi\ (a + b)} + \Delta N_z'\right)$ and $(qak + \Delta N_\theta')$. This is necessary because the forces in the buckled shell are the prebuckling forces plus the forces due to buckling. The ΔN_z, $\Delta N_z'$, ΔN_θ, and $\Delta N_\theta'$ are the forces due to buckling which are later to be expressed in terms of displacements.

All forces, moments, and twists other than the prebuckling forces are considered to be small quantities resulting from the buckling. The displacements u, v and w, and their derivatives, are also small quantities resulting from the buckling. In equations (24) to (28) products of any two such small quantities are neglected. Equation (26) is solved for Q_θ and equation (27) for Q_z. The results are substituted into equations (23), (24) and (25). This gives:

$$\Sigma F_z = 0 = a \frac{\partial (\Delta N_z)}{\partial z} + \frac{\partial N_{\theta z}}{\partial \theta} - qa (1 - k) \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right) + a Z \tag{29}$$

$$\Sigma F_\theta = 0 = a \frac{\partial N_{z\theta}}{\partial z} + \frac{\partial (\Delta N_\theta)}{\partial \theta} + \frac{\partial M_{z\theta}}{\partial z} - \frac{1}{a} \frac{\partial M_\theta}{\partial \theta} - N_z \frac{\partial^2 w}{\partial z \partial \theta} + a \theta \tag{30}$$

$$\Sigma F_r = 0 = -a \frac{\partial^2 M_z}{\partial z^2} - \frac{\partial^2 M_{\theta z}}{\partial \theta \partial z} + \frac{\partial^2 M_{z\theta}}{\partial z \partial \theta} - \frac{1}{a} \frac{\partial^2 M_\theta}{\partial \theta^2} + a N_z \frac{\partial^2 u}{\partial z^2} - qa (1 - k)$$

$$\left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right) + \Delta N_\theta + a R \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right) = 0 \tag{31}$$

(These equations are for the outer facing. A similar set is obtained for the inner facing.) Into equations (29), (30), and (31) expressions for the forces, moments and twists in terms of the displacements (ref. 3) are substituted. The surface forces

$R = qa - (q\frac{a}{r}k + \Delta\sigma_r)_{r=a}$ (for outer facing) where $q\frac{a}{r}k$ is the prebuckling stress and $\Delta\sigma_r$ the stress due to buckling,

$\theta = -(\tau_{r\theta})_{r=a}$ (for outer facing), and

$Z = -(\tau_{rz})_{r=a}$ (for outer facing),

are also expressed in terms of u, v and w and substituted into the three equations.

This leads to three equations in terms of u, v, and w for the outer facing and three similar equations for the inner facing. The equations for the outer facing are:

$$\Sigma F_z = 0 = a^2 \frac{\partial^2 w}{\partial z^2} + \frac{1-\mu}{2} \frac{\partial^2 w}{\partial \theta^2} + a \frac{1+\mu}{2} \frac{\partial^2 v}{\partial z \partial \theta} + a\mu \frac{\partial u}{\partial z} - a\,\phi_1 (1-k) \left(\frac{\partial^2 v}{\partial z \partial \theta} + \frac{\partial u}{\partial z} \right)$$

$$- a^2 \frac{(1-\mu^2)}{Et} G_{rz} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \tag{32}$$

$$\Sigma F_\theta = 0 = \frac{1+\mu}{2} a \frac{\partial^2 w}{\partial z \partial \theta} + (1+\alpha) \frac{\partial^2 v}{\partial \theta^2} + \frac{1-\mu}{2} a^2 \frac{\partial^2 v}{\partial z^2} + \alpha (1-\mu) a^2 \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial \theta} - a^2 \alpha$$

$$\frac{\partial^3 u}{\partial^2 z \partial \theta} - \alpha \frac{\partial^3 u}{\partial \theta^3} - a\,\phi_2 \frac{\partial^2 w}{\partial z \partial \theta} - a^2 \frac{1-\mu^2}{Et} G_{r\theta} \left[\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right] \tag{33}$$

$$\Sigma F_r = 0 = -a\mu \frac{\partial w}{\partial z} - \frac{\partial v}{\partial \theta} - u + \alpha \frac{\partial^3 v}{\partial \theta^3} + (2-\mu)\,\alpha a^2 \frac{\partial^3 v}{\partial \theta \partial z^2} - \alpha a^4 \frac{\partial^4 u}{\partial z^4} - \alpha \frac{\partial^4 u}{\partial \theta^4}$$

$$- 2\alpha a^2 \frac{\partial^4 u}{\partial \theta^2 \partial z^2} + a^2 \phi_2 \frac{\partial^2 u}{\partial z^2} - a\,\phi_1 (1-k) \left(1 - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \right) -$$

$$a^2 \frac{(1-\mu^2)}{Et} \left(qa - aq\,k - E_c \frac{\partial u}{\partial r} \right) \left(1 + \frac{1}{a} \frac{\partial v}{\partial \theta} + \frac{u}{a} + \frac{\partial w}{\partial z} \right). \tag{34}$$

To achieve proper interaction, between the core and the facings, the displacements of the middle surfaces of the facings are set equal to the displacements of the core at r = a and r = b.

Thus displacements u, v and w in equations (32), (33), and (34) are replaced by equations (20), (21), and (22) with r made equal to a. In this manner three equations in six arbitrary constants (A, B, C, D, L, and F) are written for the outer facing. In a like fashion three equations are written for the inner facing. The coefficients of the six arbitrary constants are shown in the form of a determinant on the following page.

$a\ [-\mu\lambda + \lambda\phi_1 (1-k)] -$ $na\ [-\frac{1+\mu}{2} n\lambda + n\lambda\phi_1 (1-k)] +$ $\lambda a\ [-\lambda^2 - \frac{1-\mu}{2} n^2]$	$a\ [-\mu\lambda + \lambda\phi_1 (1-k)] +$ $\frac{a\lambda}{2} [-\lambda^2 - \frac{1-\mu}{2} n^2] +$ $\frac{a}{\lambda}\beta$	$a\ [-\mu\lambda + \lambda\phi_1 (1-k)] +$ $\frac{a}{n}\delta_{n0} [-\frac{1+\mu}{2} n\lambda + n\lambda\phi_1 (1-k)]$
$a\ [-\mu\lambda \frac{b}{a} + \lambda \frac{b}{a} \phi_1 k] -$ $na\ [-\frac{1+\mu}{2} n\lambda \frac{b}{a} + n\lambda \frac{b}{a} \phi_1 k] +$ $\lambda b\ [-\lambda^2 \frac{b^2}{a^2} - \frac{1-\mu}{2} n^2]$	$b\ [-\mu\lambda \frac{b}{a} + \lambda \frac{b}{a} \phi_1 k] +$ $nb \log \frac{b}{a} [-\frac{1+\mu}{2} n\lambda \frac{b}{a} + n\lambda \frac{b}{a} \phi_1 k] +$ $a\lambda [\frac{b^2}{2a^2} - \frac{\delta_s}{\lambda^2} \log \frac{b}{a}] [-\lambda^2 \frac{b^2}{a^2} - \frac{1-\mu}{2} n^2] - \frac{b}{\lambda}\beta$	$\frac{a^2}{b} [-\mu\lambda \frac{b}{a} + \lambda \frac{b}{a} \phi_1 k] +$ $\frac{1}{n} \frac{a^2}{b} \delta_{n0} [-\frac{1+\mu}{2} n\lambda \frac{b}{a} + n\lambda \frac{b}{a} \phi_1 k] +$ $\lambda a \log \frac{b}{a} [-\lambda^2 \frac{b^2}{a^2} - \frac{1-\mu}{2} n^2]$
$a\ [-n - an^3 - an\lambda^2] -$ $na\ [-\frac{1-\mu}{2} \lambda^2 - n^2 - an^2 - (1-\mu) a\lambda^2] +$ $\lambda a\ [-\frac{1+\mu}{2} n\lambda + \lambda n\phi_2]$	$a\ [-n - an^3 - an\lambda^2] +$ $\frac{a\lambda}{2} [-\frac{1+\mu}{2} n\lambda + \lambda n\phi_2]$	$a\ [-n - an^3 - an\lambda^2] +$ $\frac{a}{n}\delta_{n0} [-\frac{1-\mu}{2} \lambda^2 - n^2 - an^2 (1-\mu) a\lambda^2] +$ $\frac{a}{n}\beta$
$a\ [-n - a'n^3 - a'\lambda^2 \frac{b^2}{a^2} n] -$ $na\ [-\frac{1-\mu}{2} \lambda^2 \frac{b^2}{a^2} - n^2 - a'n^2 - (1-\mu) a'\lambda^2 \frac{b^2}{a^2}] +$ $\lambda b\ [-\frac{1+\mu}{2} n\lambda \frac{b}{a} + \lambda \frac{b}{a} n\phi_2]$	$b\ [-n - a'n^3 - a'\lambda^2 \frac{b^2}{a^2} n] +$ $nb \log \frac{b}{a} [-\frac{1-\mu}{2} \lambda^2 \frac{b^2}{a^2} - n^2 - a'n^2 - (1-\mu) a'\lambda^2 \frac{b^2}{a^2}] +$ $a\lambda [\frac{b^2}{2a^2} - \frac{\delta_s}{\lambda^2} \log \frac{b}{a}] [-\frac{1+\mu}{2} n\lambda \frac{b}{a} + \lambda \frac{b}{a} n\phi_2]$	$\frac{a^2}{b} [-n - a'n^3 - a'\lambda^2 \frac{b^2}{a^2} n] +$ $\frac{1}{n} \frac{a^2}{b} \delta_{n0} [-\frac{1-\mu}{2} \lambda^2 \frac{b^2}{a^2} - n^2 - a'n^2 - (1-\mu) a'\lambda^2 \frac{b^2}{a^2}] +$ $\lambda a \log \frac{b}{a} [-\frac{1+\mu}{2} n\lambda \frac{b}{a} + \lambda \frac{b}{a} n\phi_2] - \frac{a}{n}\beta$
$a\ [-1 - a\lambda^4 - an^4 - 2a\lambda^2 n^2 - \lambda^2\phi_2 + (1-k)\phi_1 - (1-k) n^2\phi_1] -$ $na\ [-n - an^3 - (2-\mu) an\lambda^2] +$ $\lambda a\ [-\mu\lambda]$	$a\ [-1 - a\lambda^4 - an^4 - 2a\lambda^2 n^2 - \lambda^2\phi_2 + (1-k)\phi_1 - (1-k) n^2\phi_1] +$ $\frac{a\lambda}{2} [-\mu\lambda] -$ $a\beta$	$a\ [-1 - a\lambda^4 - an^4 - 2a\lambda^2 n^2 - \lambda^2\phi_2 + (1-k)\phi_1 - (1-k) n^2\phi_1] +$ $\frac{a}{n}\delta_{n0} [-n - an^3 - (2-\mu) an\lambda^2] +$ $a\beta$
$a\ [-1 - a'\lambda^4 \frac{b^4}{a^4} - a'n^4 - 2a'\lambda^2 \frac{b^2}{a^2} n^2 - \lambda^2 \frac{b^2}{a^2} \phi_2 + k\phi_1 - kn^2\phi_1] -$ $na\ [-n - a'n^3 - (2-\mu) a'n\lambda^2 \frac{b^2}{a^2}] +$ $\lambda b\ [-\mu\lambda \frac{b}{a}]$	$b\ [-1 - a'\lambda^4 \frac{b^4}{a^4} - a'n^4 - 2a'\lambda^2 \frac{b^2}{a^2} n^2 + k\phi_1 - kn^2\phi_1] +$ $nb \log \frac{b}{a} [-n - a'n^3 - (2-\mu) a'n\lambda^2 \frac{b^2}{a^2}] -$ $a\lambda [\frac{b^2}{2a^2} - \frac{\delta_s}{\lambda^2} \log \frac{b}{a}] [-\mu\lambda \frac{b}{a}] +$ $+ \frac{b^2}{a}\beta$	$\frac{a^2}{b} [-1 - a'\lambda^4 \frac{b^4}{a^4} - a'n^4 - 2a'\lambda^2 \frac{b^2}{a^2} n^2 - \lambda^2 \frac{b^2}{a^2} \phi_2 + k\phi_1 - kn^2\phi_1] +$ $\frac{1}{n} \frac{a^2}{b} \delta_{n0} [-n - a'n^3 - (2-\mu) a'n\lambda^2 \frac{b^2}{a^2}] +$ $\lambda a \log \frac{b}{a} [-\mu\lambda \frac{b}{a}] -$ $a\beta$

1

	$a\ [-\mu\lambda + \lambda\phi_1(1-k)] +$ $\frac{a}{n}\delta_{n0}[-\frac{1+\mu}{2}n\lambda + n\lambda\phi_1(1-k)]$	$-an[-\frac{1+\mu}{2}n\lambda + n\lambda\phi_1(1-k)] -$ $\lambda a[-\lambda^2 - \frac{1-\mu}{2}n^2]$	$a[-\frac{1+\mu}{2}n\lambda + n\lambda\phi_1(1-k)]$	$a[-\lambda^2 - \frac{1-\mu}{2}n^2]$
	$\frac{a^2}{b}[-\mu\lambda\frac{b}{a} + \lambda\frac{b}{a}\phi_1 k] +$ $\frac{1}{n}\frac{a^2}{b}\delta_{n0}[-\frac{1+\mu}{2}n\lambda\frac{b}{a} + n\lambda\frac{b}{a}\phi_1 k] +$ $\lambda a\log\frac{b}{a}[-\lambda^2\frac{b^2}{a^2} - \frac{1-\mu}{2}n^2]$	$a\log\frac{b}{a}[-\mu\lambda\frac{b}{a} + \lambda\frac{b}{a}\phi_1 k] -$ $an(\log\frac{b}{a}+1)[-\frac{1+\mu}{2}n\lambda\frac{b}{a} + n\lambda\frac{b}{a}\phi_1 k] +$ $\lambda b(\log\frac{b}{a}-1)[-\lambda^2\frac{b^2}{a^2} - \frac{1-\mu}{2}n^2]$	$b[-\frac{1+\mu}{2}n\lambda\frac{b}{a} + n\lambda\frac{b}{a}\phi_1 k]$	$a[-\lambda^2\frac{b^2}{a^2} - \frac{1-\mu}{2}n^2]$
	$a[-n - \alpha n^3 - \alpha n\lambda^2] +$ $\frac{a}{n}\delta_{n0}[-\frac{1-\mu}{2}\lambda^2 - n^2 - \alpha n^2(1-\mu)\alpha\lambda^2] +$ $\frac{a}{n}\beta$	$-an[-\frac{1-\mu}{2}\lambda^2 - n^2 - \alpha n^2 - (1-\mu)\alpha\lambda^2] -$ $\lambda a[-\frac{1+\mu}{2}n\lambda + \lambda n\phi_2]$	$a[-\frac{1-\mu}{2}\lambda^2 - n^2 - \alpha n^2 - (1-\mu)\alpha\lambda^2]$	$a[-\frac{1+\mu}{2}n\lambda + \lambda n\phi_2]$
$\frac{b^2}{a^2}] +$	$\frac{a^2}{b}[-n - \alpha n^3 - \alpha\lambda^2\frac{b^2}{a^2}n] +$ $\frac{1}{n}\frac{a^2}{b}\delta_{n0}[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2} - n^2 - \alpha n^2 - (1-\mu)\alpha\lambda^2\frac{b^2}{a^2}] +$ $\lambda a\log\frac{b}{a}[-\frac{1+\mu}{2}n\lambda\frac{b}{a} + \lambda\frac{b}{a}n\phi_2] - \frac{a}{n}\beta$	$a\log\frac{b}{a}[-n-\alpha n^3 - \alpha\lambda^2\frac{b^2}{a^2}n] -$ $an(\log\frac{b}{a}+1)[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2} - n^2 - \alpha n^2 - (1-\mu)\alpha\lambda^2\frac{b^2}{a^2}] +$ $\lambda b(\log\frac{b}{a}-1)[-\frac{1+\mu}{2}n\lambda\frac{b}{a} + \lambda\frac{b}{a}n\phi_2]$	$b[-\frac{1-\mu}{2}\lambda^2\frac{b^2}{a^2} - n^2 - \alpha n^2 - (1-\mu)\alpha\lambda^2\frac{b^2}{a^2}]$	$a[-\frac{1+\mu}{2}n\lambda\frac{b}{a} + \lambda\frac{b}{a}n\phi_2]$
$-k)n^2\phi_1] +$	$a[-1 - \alpha\lambda^4 - \alpha n^4 - 2\alpha\lambda^2 n^2 - \lambda^2\phi_2 + (1-k)\phi_1 - (1-k)n^2\phi_1] +$ $\frac{a}{n}\delta_{n\theta}[-n - \alpha n^3 - (2-\mu)\alpha n\lambda^2] +$ $a\beta$	$-an[-n-\alpha n^3 - (2-\mu)\alpha n\lambda^2] -$ $\lambda a[-\mu\lambda] -$ $a\beta$	$a[-n-\alpha n^3-(2-\mu)\alpha n\lambda^2]$	$a[-\mu\lambda]$
$\phi_1] +$	$\frac{a^2}{b}[-1-\alpha\lambda^4\frac{b^4}{a^4} - \alpha n^4 - 2\alpha\lambda^2\frac{b^2}{a^2}n^2 - \lambda^2\frac{b^2}{a^2}\phi_2 + k\phi_1 - kn^2\phi_1] +$ $\frac{1}{n}\frac{a^2}{b}\delta_{n\theta}[-n-\alpha n^3 - (2-\mu)\alpha n\lambda^2\frac{b^2}{a^2}] +$ $\lambda a\log\frac{b}{a}[-\mu\lambda\frac{b}{a}]$ $a\beta$	$a\log\frac{b}{a}[-1-\alpha\lambda^4\frac{b^4}{a^4} - \alpha n^4 - 2\alpha\lambda^2\frac{b^2}{a^2}n^2 - \lambda^2\frac{b^2}{a^2}\phi_2 + k\phi_1 - kn^2\phi_1] -$ $an(\log\frac{b}{a}+1)[-n-\alpha n^3-(2-\mu)\alpha n\lambda^2\frac{b^2}{a^2}] +$ $\lambda b(\log\frac{b}{a}-1)[-\mu\lambda\frac{b}{a}] +$ $b\beta$	$b[-n-\alpha n^3-(2-\mu)\alpha n\lambda^2\frac{b^2}{a^2}]$	$a[-\mu\lambda\frac{b}{a}]$

2

It is possible to find simultaneous values of ϕ_1 and ϕ_2 for which these six equations will be satisfied for any values of the arbitrary constants. Mathematically this means that for such a combination of loads the deflections are indeterminate. The shell becomes elastically unstable and the loads that bring about this condition are called critical loads.

Numerical Computations

A literal solution of the sixth order determinant for the eigenvalues is not feasible. A numerical solution, from which curves may be drawn, is possible if a digital computer is used. A CPC Model 2 was available to make computations. Even with the CPC the task seemed overwhclming. If, however, E_c is made infinite, some of thc terms of determinant vanish. The assumption that E_c is infinite is common in work with sandwich construction and has been found to give satisfactory results in most cases. The sixth order determinant with E_c made infinite is represented below:

$$
\begin{vmatrix}
A_1 & B_1 & C_1 & 0 & F_1 & L_1 \\
A_2 & B_2 & C_2 & 0 & F_2 & L_2 \\
A_3 & 0 & C_3 & 0 & F_3 & L_3 \\
A_4 & B_4 & C_4 & 0 & F_4 & L_4 \\
A_5 & B_5 & C_5 & -\gamma & F_5 & L_5 \\
A_6 & B_6 & C_6 & +\frac{b}{a}\gamma & F_6 & L_6
\end{vmatrix}
$$

This determinant is then reduced to a fourth order determinant shown below:

$$
\begin{vmatrix}
A_1 C_3 - C_1 A_3 & B_1 & C_1 L_3 - L_1 C_3 & F_1 L_3 - L_1 F_3 \\
A_2 C_3 - C_2 A_3 & B_2 & C_2 L_3 - L_2 C_3 & F_2 L_3 - L_2 F_3 \\
A_4 C_3 - C_4 A_3 & B_4 & C_4 L_3 - L_4 C_3 & F_4 L_3 - L_4 F_3 \\
(A_5 \frac{b}{a} + A_6) C_3 - & B_5 \frac{b}{a} + & (C_5 \frac{b}{a} + C_6) L_3 - & (F_5 \frac{b}{a} + F_6) L_3 - \\
(C_5 \frac{b}{a} + C_6) A_3 & B_6 & (L_5 \frac{b}{a} + L_6) C_3 & (L_5 \frac{b}{a} + L_6) F_3
\end{vmatrix}
$$

The determinant is then programmed for the CPC. A trial and error solution is made by substituting values of ϕ_1 or ϕ_2 until a value is found that will make the determinant zero. This was done by finding values on each side of zero and interpolating to find the eigenvalue.

Discussion of Results

Since the problem is solved by numerical methods the results are presented by the curves shown in figures 6, 7, 8 and 9. The values of $\frac{b}{a} = 0.97$ and $\frac{a}{t} = 1,000$ were used for all of the curves.

Figure 6 is a family of curves in which $-\phi_2$ is plotted against $\frac{\ell}{ma}$ for different values of n. In these curves the values $\frac{E}{G_{r\theta}} = 10,000$ and $\frac{E}{G_{rz}} = 1,000$ were used. Such a set of curves is used in the following manner.

Knowing $\frac{\ell}{a}$ of the cylinder one picks a value for m and n. A value of $-\phi_2$ is determined by reading above $\frac{\ell}{ma}$ on the corresponding n curve. This procedure is repeated until the lowest possible value of $-\phi_2$ is found. The axial load under which the cylinder will buckle can then be determined.

The curves of figure 7 differ from those of figure 6 as a result of making α and α' zero. This is equivalent to neglecting the bending stiffnesses of the facings. A comparison of the curves of figure 7 with those of figure 6 shows that for values of $\frac{\ell}{ma}$ greater than 0.15 there is little difference. It can be concluded that only for very short cylinders need the bending stiffnesses of the facings be considered. For $\frac{\ell}{ma}$ less than 0.15 the curves of figure 7 approach

$$-\phi_2 = \frac{G_{rz}\, a\, (1 - \mu^2)\, (1 - \frac{b}{a})^2}{Et \log \frac{b}{a}\, (1 + \frac{b}{a})} \; .$$

This value is obtained by making n and ℓ zero and expanding the determinant. Solving for N_z and replacing $\log \frac{b}{a}$ by the first term of its series expansion shows that

$$N_z = - \frac{(a - b)}{1 + \frac{b}{a}} \; G_{rz} \; .$$

The curves of figure 8 are the result of increasing G_{rz} and $G_{r\theta}$ tenfold. The value of $-\phi_2$ corresponding to

$$N_z = - \frac{(a - b)}{1 + \frac{b}{a}} \; G_{rz}$$

appears as a flattening of the curve in the region of $\frac{\ell}{ma} = 0.01$. For smaller values of $\frac{\ell}{ma}$ the curve rises due to the stiffness of the facings. For values of $\frac{\ell}{ma}$ greater than 0.1 the curves show a considerably lower buckling load.

From a comparison of figures 6 and 8 it appears that as G_{rz} is decreased the buckling load for all cylinders, except those long enough to fail as an Euler column, will approach

$$N_z = - \frac{(a - b)}{1 + \frac{b}{a}} \; G_{rz}$$

This limit has been recognized (ref. 4) as the critical load for shells with a low value of G_{rz}. It should be noted that this load depends only upon the thickness and the modulus of rigidity of the core.

Figure 9 shows curves of $-\phi_1$ plotted against $\frac{\ell}{ma}$ for different values of $n_1\phi_2$ for these curves was taken to be

$$\phi_2 = \frac{\phi_1}{2\left(1 + \frac{b}{a}\right)}$$

This represents the case for an end load equal to $q\pi a^2$. The situation is like that of a cylinder, with ends, under uniform pressure. The ends of course stiffen the cylinder, but, if the cylinder is not too short, reasonable results can be expected. Since ϕ_1 decreases as $\frac{\ell}{ma}$ is increased it must be concluded that the cylinder will buckle with $m = 1$. The critical pressure can be determined by reading ϕ_1 from the lowest n curve.

Conclusions

Although only a few curves were drawn it is apparent that this analysis is helpful in understanding the effect produced by a variation of the parameters that enter the problem. Further study is required before it will be known whether the actual buckling load may be predicted.

It is felt that the method by which this problem is solved can be applied with advantage to many problems of sandwich construction. Unfortunately in most cases a numerical solution will be required.

References

(1) Raville, M. E. "Analysis of Long Cylinders of Sandwich Construction under Uniform External Lateral Pressure" (in: Forest Products Laboratory Report No. 1844. Nov. 1954).

(2) Osgood, W. R., and Joseph, J. A. "On the General Theory of Thin Shells" (in: Journal of Applied Mechanics, Vol. 17, No. 4, Dec. 1950).

(3) Timoshenko, S. "Theory of Elastic Stability" 1st ed., N. Y. 1936.

(4) Teichmann, F. K., Wang, C., Gerard, G. "Buckling of Sandwich Cylinders Under Axial Compression" (in: Journal of the Aeronautical Sciences, Vol. 18, No. 6, June 1951).

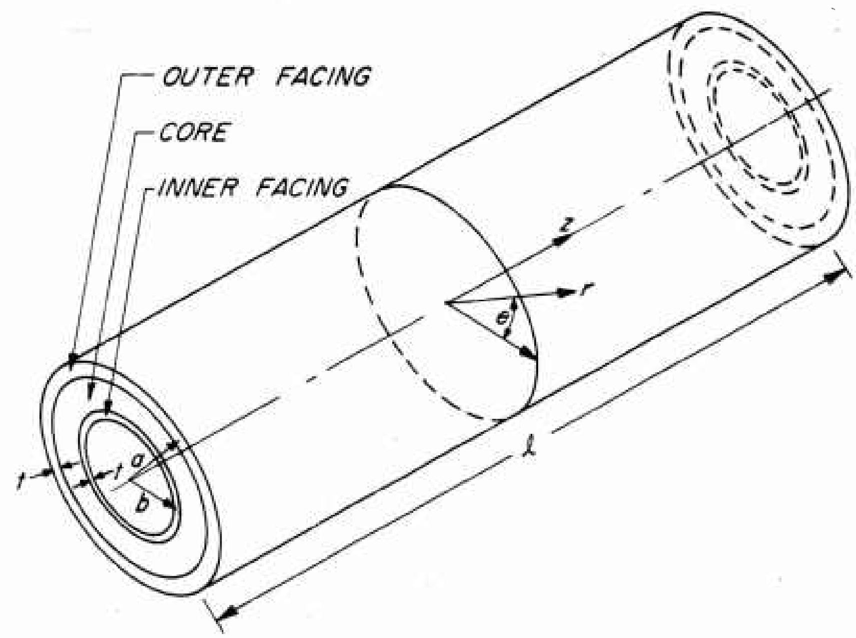

Figure 1. --Sandwich cylinder.

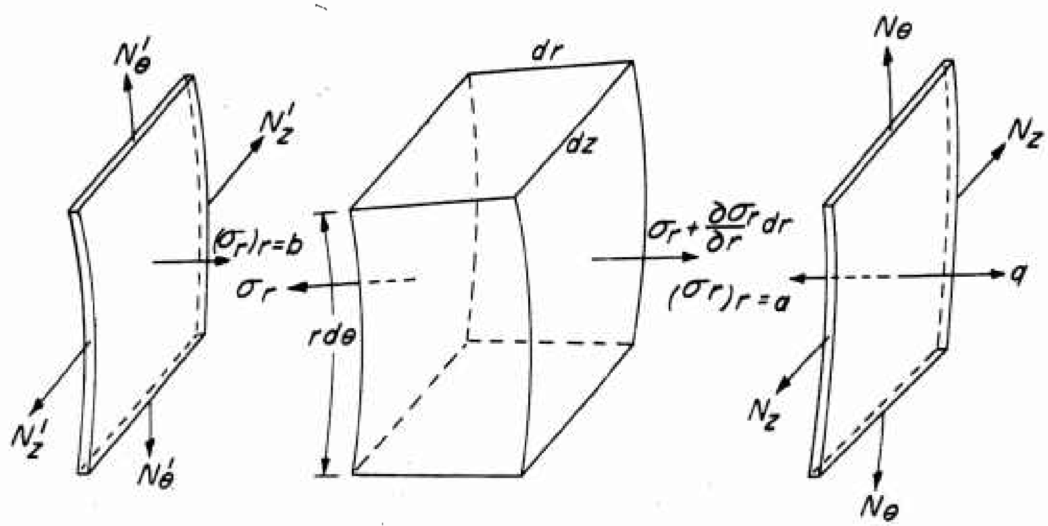

Figure 2. --Differential elements of core and facings before buckling.

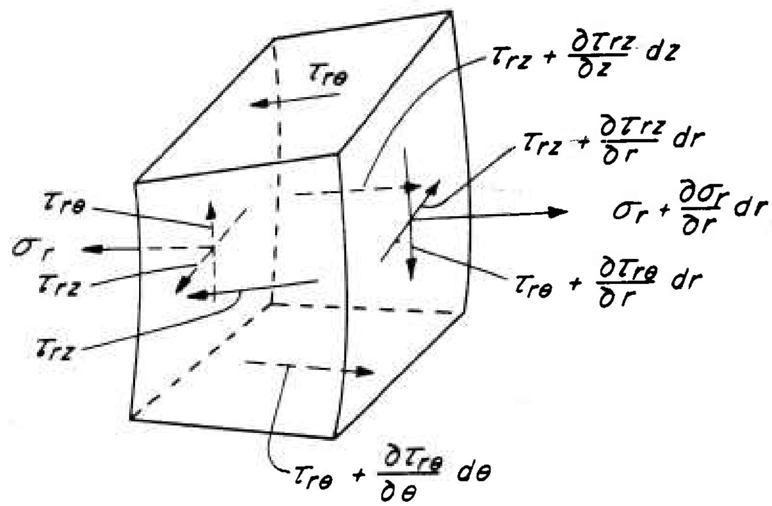

Figure 3. --Differential element of core.

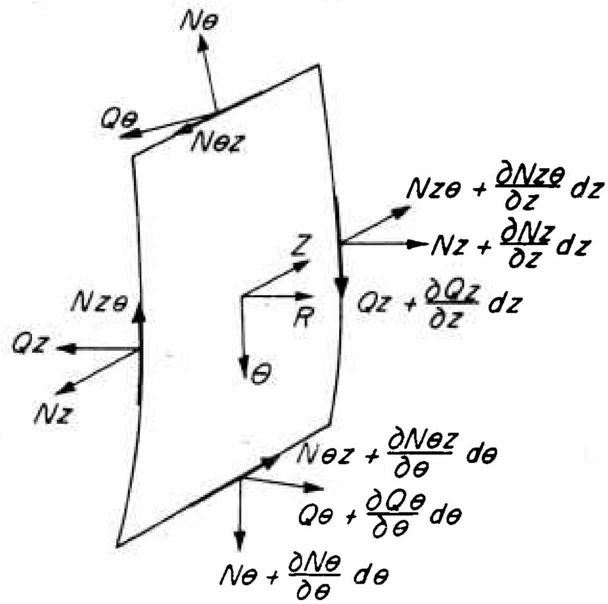

Figure 4. --Differential element of facing showing forces.

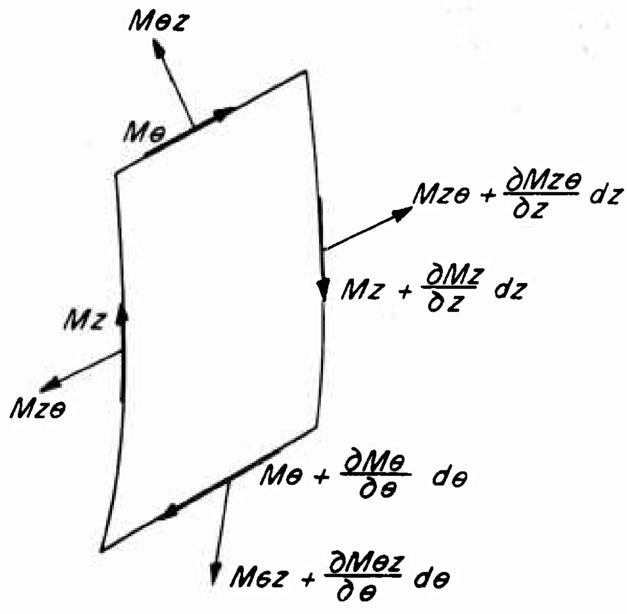

Figure 5. --Differential element of facing showing moments and twists.

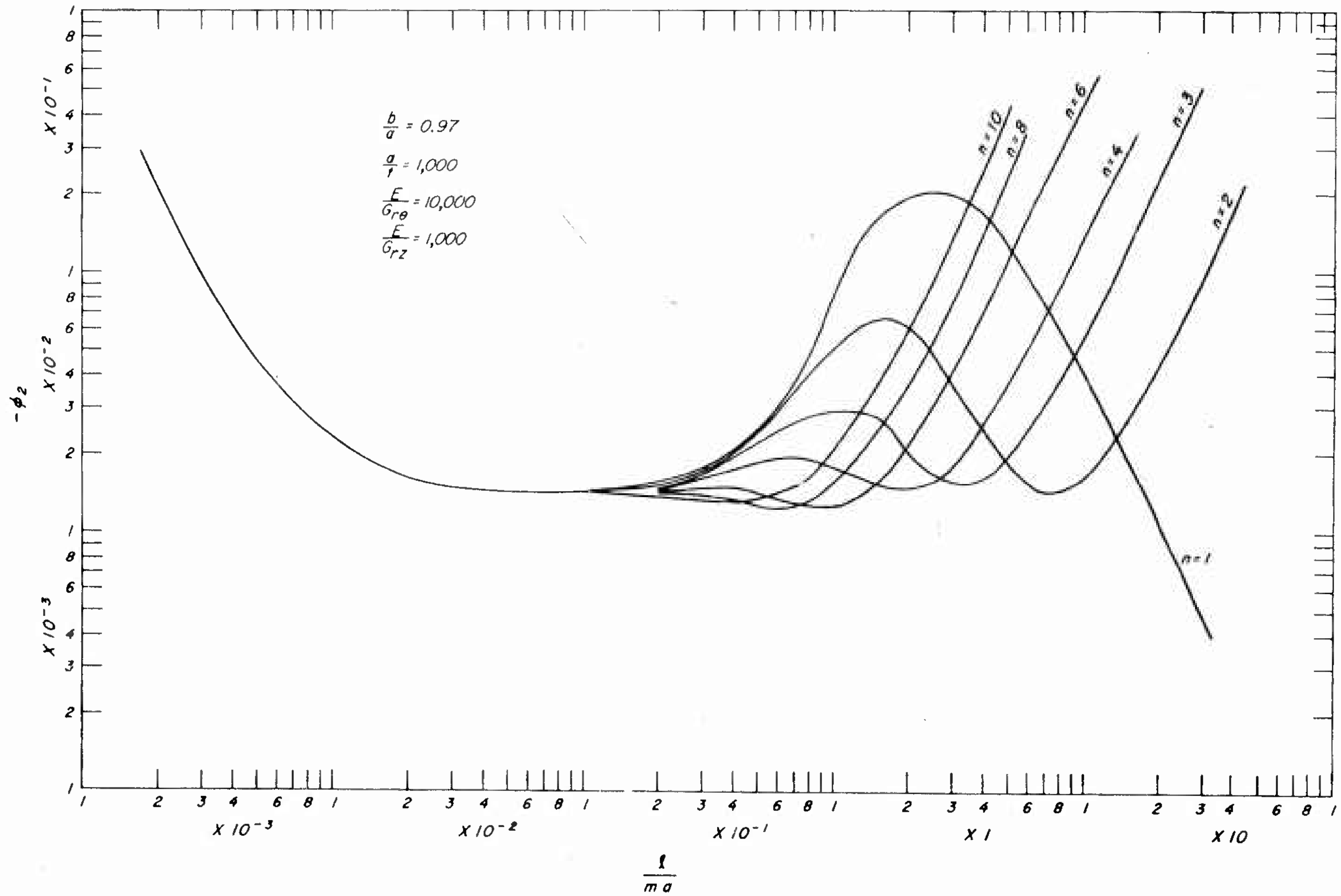

Figure 6. --Critical axial load in terms of ϕ_2 versus $\frac{\ell}{ma}$

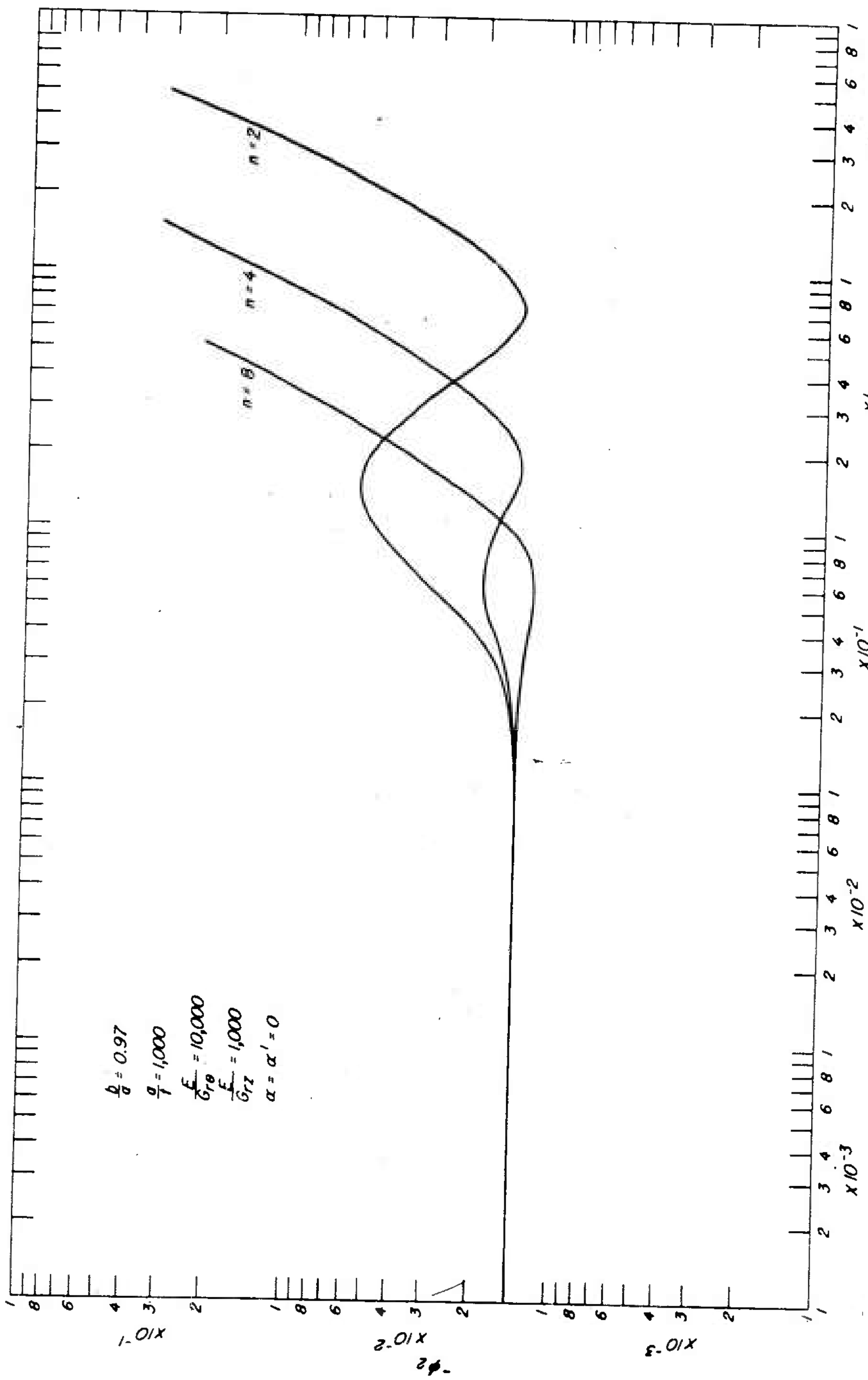

Figure 7. --Critical axial load in terms of ϕ_2 versus $\frac{\ell}{ma}$

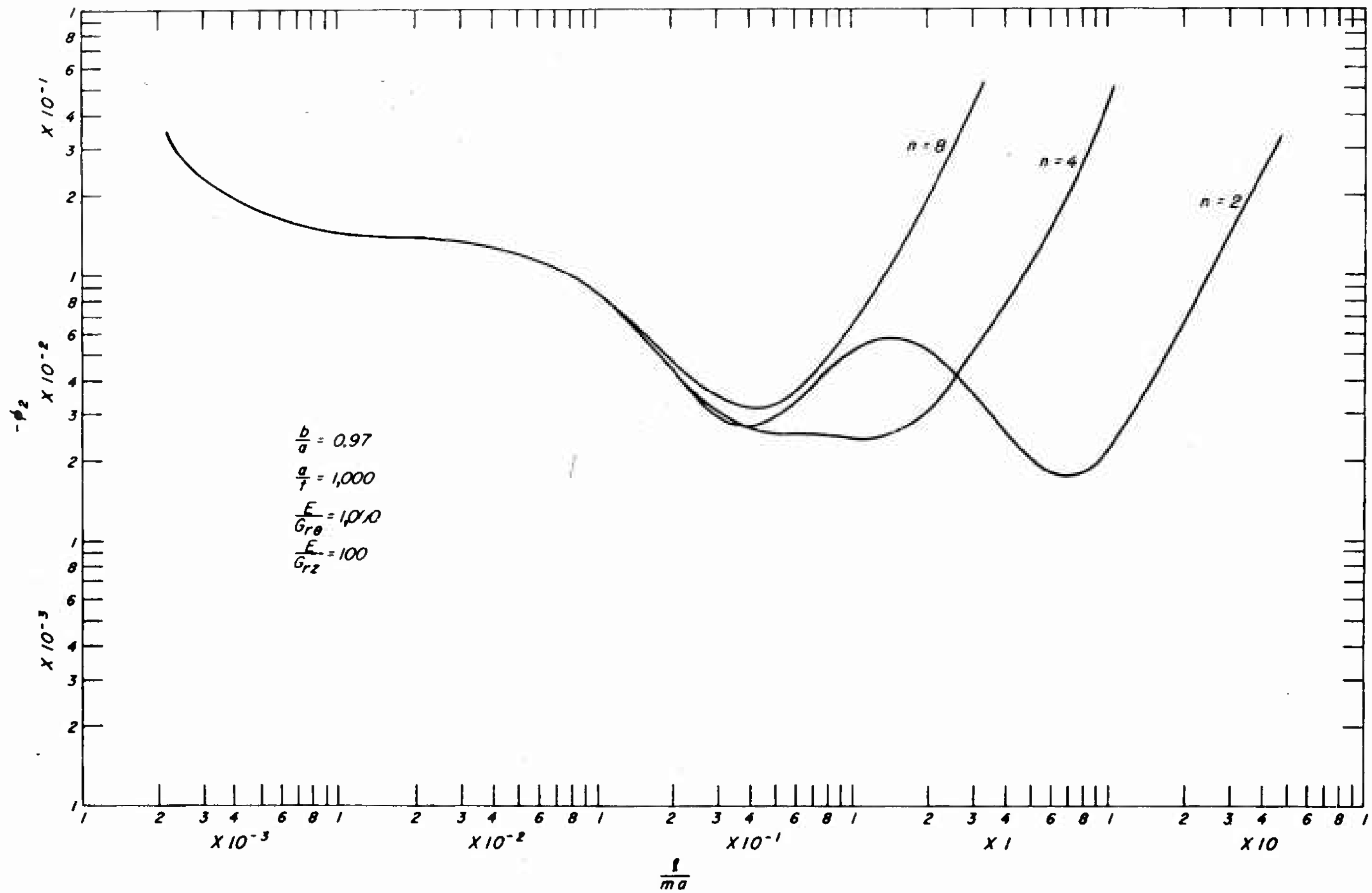

Figure 8. --Critical axial load in terms of ϕ_2 versus $\frac{\ell}{ma}$

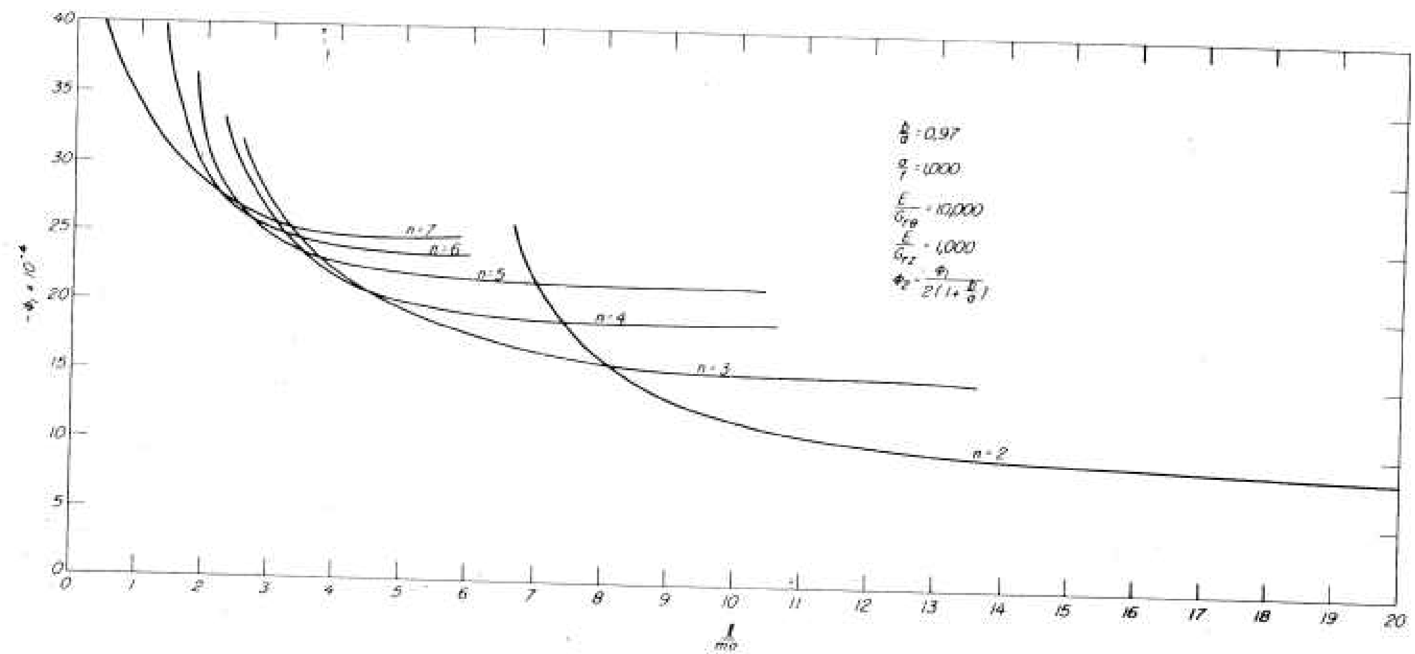

Figure 9. --Critical pressure in terms of ϕ_1 versus $\frac{l}{ma}$

SECTION II

BUCKLING OF CYLINDERS OF SANDWICH CONSTRUCTION IN AXIAL COMPRESSION[1]

By

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Summary

This report presents a theoretical analysis for the behavior of long, circular, cylindrical shells of sandwich construction under axial compressive loads. The analysis is designed to evaluate the effects of the relatively low shearing moduli of sandwich cores on buckling stresses. Families of curves are presented for use in designing shells of sandwich construction having isotropic facings and orthotropic or isotropic cores.

The results of the theoretical analysis were compared with those obtained from tests on a series of curved panels. It was found that the theory applied reasonably well to curved plates of sizes sufficient to include at least one ideal buckle. Application of the theory thus is not limited to long, complete cylinders.

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[1] This progress report is one of a series prepared and distributed by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics Order No. NAer 01237 and 01202, and U. S. Air Force No. USAF 18 (600)-70. Results here reported are preliminary and may be revised as additional data become available. Original report published June 1952.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Introduction

In the design of aircraft and guided missiles, it was found necessary to devise a method of determining the stress at which curved sandwich panels subjected to axial compression become elastically unstable. It is known that, for thin, homogeneous materials, a curved form greatly increases the critical load as compared to a flat sheet of the same approximate size. A similar increase may be expected for curved sandwich panels. Although this report applies primarily to sandwich construction for aircraft, the results are general and apply to any structures of the type considered.

This report presents a theoretical analysis of the behavior of long, circular, cylindrical shells of sandwich construction under axial compressive loads and an experimental confirmation of this analysis by tests on curved panels of sufficient size to include at least one ideal buckle. Thus, these panels are assumed to simulate the action of complete cylinders.

The buckling of a homogeneous, isotropic, thin-walled cylinder was treated by von Karman and Tsien (16)[3] and by Tsien (13, 14) in related papers. These authors assumed, in addition to the wave form of the classical theory, inward buckles of diamond shape to represent the characteristic buckles that are actually observed. They used an energy method to determine the critical compressive stress. This method, in which only diamond-shaped buckles are used, was applied by March (7) to cylinders made of plywood, an orthotropic material. Particular attention was paid to the effect of initial irregularities that contribute to the observed scatter of experimentally determined critical stresses of both isotropic and orthotropic cylinders.

In this report, the effect of shear deformation in the core of a sandwich cylinder is taken into account by employing an approximate "tilting" method. This method was used by Williams, Leggett, and Hopkins in their analysis of flat sandwich panels (18) and by Leggett and Hopkins in their analysis of flat sandwich panels and cylinders (4). It amounts essentially to assuming that the transverse components of shear stress are constant across the thickness of the core. The form of buckles assumed by Leggett and Hopkins (4) in the cylinder is different from that assumed in this report.

The core and facings are taken to be orthotropic, with two of their natural axes parallel, respectively, to the axial and circumferential directions of the cylinder. The facings, which may be equal or unequal in thickness, are

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[3] Underlined numbers in parentheses refer to Literature Cited at end of report.

assumed to be thin, but their flexural rigidities are not neglected, as these may be of importance in certain cases. All stress components in the core are neglected except the transverse shear components. It was pointed out by Reissner (12) that the stress component in the core normal to the facings may be of importance in the analysis of sandwich shells. Preliminary calculations indicate that the effect of this component is small in the problem under consideration.

As was done in work described by Forest Products Laboratory Report No. 1322-A (7), initial irregularities are assumed to be present and to grow under increasing compressive load until buckling occurs. For a discussion of this important matter, the reader is referred to that report and, in particular, to the observations of the growth of artificially produced initial irregularities. Also as described in report No. 1322-A, a large deflection theory is used to take into account the nonlinear support associated with the curvature of the shell, as discussed by von Karman, Dunn, and Tsien (17). The derivations of the differential equation for a stress function and of the expression for the energy of deformation are extensions of the analysis used by von Karman and Tsien (16) for the homogeneous, isotropic cylinder to the sandwich cylinder composed of orthotropic materials. Suitable modification is made for the effect of shear deformation in the core of the sandwich.

Theoretical Analysis

Choice of Axes Notation

The choice of axes is shown in figure 1, the coordinate y being measured along the circumference. The notations for stress and strain are those of Love's treatise (5). The components of displacement in the axial, circumferential, and radial directions, respectively, are u, v, and w, the latter being positive inward. Since initial irregularities of the cylindrical surface are assumed, the symbol w_o is used to denote the initial distance, measured radially, of a point of the middle surface from a true cylindrical surface of radius r, and the symbol w to denote the corresponding distance at any stage of the deformation. The thickness of the core is denoted by c and that of each facing by f_1 and f_2, respectively.

Extensional Strains and Stresses

Expressions can now be written for the extensional strains uniform across the thickness of the cylindrical shell and for the corresponding mean membrane stresses. On these will be superposed a system of flexural strains, and the energy of deformation associated with each system of strains will be found.

The extensional strains are expressed by the equations:

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_o}{\partial x}\right)^2$$

$$e_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{1}{2}\left(\frac{\partial w_o}{\partial y}\right)^2 - \frac{w}{r} + \frac{w_o}{r} \qquad (1)$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w_o}{\partial x}\frac{\partial w_o}{\partial y}$$

In each facing, the corresponding stress components are:

$$X_x = \frac{E_x}{\lambda}\left(e_{xx} + \sigma_{yx}\, e_{yy}\right)$$

$$Y_y = \frac{E_y}{\lambda}\left(e_{yy} + \sigma_{xy}\, e_{xx}\right) \qquad (2)$$

$$X_y = \mu_{xy}\, e_{xy}$$

where $\underline{E_x}$ and $\underline{E_y}$ are Young's moduli, μ_{xy} is the modulus of rigidity for shearing strains referred to the $\underline{x}$ and $\underline{y}$ directions, $\underline{\sigma_{xy}}$ and $\underline{\sigma_{yx}}$ are Poisson's ratios, and $\lambda = 1 - \sigma_{xy}\,\sigma_{yx}$. All of these quantities are elastic properties of the facings. Because the stress components $\underline{X_x}$, $\underline{Y_y}$, and $\underline{X_y}$ are neglected in the core, the mean membrane stress components for the cylinder are:

$$\overline{X_x} = \frac{E_a}{\lambda}\left(e_{xx} + \sigma_{yx}\, e_{yy}\right)$$

$$\overline{Y_y} = \frac{E_b}{\lambda}\left(e_{yy} + \sigma_{xy}\, e_{xx}\right) \qquad (3)$$

$$\overline{X_y} = \mu_m\, e_{xy}$$

where:

$$E_a = \frac{E_x\,(f_1 + f_2)}{h}, \quad E_b = \frac{E_y\,(f_1 + f_2)}{h}, \quad \mu_m = \frac{\mu_{xy}\,(f_1 + f_2)}{h} \qquad (4)$$

and:

$$h = c + f_1 + f_2 \tag{5}$$

From the relation:

$$E_x \sigma_{yx} = E_y \sigma_{xy} \tag{6}$$

that holds for orthotropic materials and equations (4), it follows that:

$$E_a \sigma_{yx} = E_b \sigma_{xy} \tag{7}$$

By using this relation, it is found from equation (3) that:

$$\begin{aligned} e_{xx} &= \frac{1}{E_a} \overline{X_x} - \frac{\sigma_{yx}}{E_b} \overline{Y_y} \\ e_{yy} &= \frac{1}{E_b} \overline{Y_y} - \frac{\sigma_{xy}}{E_a} \overline{X_x} = \frac{1}{E_b} \overline{Y_y} - \frac{\sigma_{xy}}{E_b} \overline{X_x} \\ e_{xy} &= \frac{1}{\mu_m} \overline{X_y} \end{aligned} \tag{8}$$

The mean membrane stress components satisfy the equations of equilibrium:

$$\begin{aligned} \frac{\partial \overline{X}_x}{\partial x} + \frac{\partial \overline{X}_y}{\partial y} &= 0 \\ \frac{\partial \overline{X}_y}{\partial x} + \frac{\partial \overline{Y}_y}{\partial y} &= 0 \end{aligned} \tag{9}$$

They can consequently be expressed in terms of a stress function as follows:

$$\overline{X}_x = \frac{\partial^2 F}{\partial y^2}, \quad \overline{Y}_y = \frac{\partial^2 F}{\partial x^2}, \quad \overline{X}_y = - \frac{\partial^2 F}{\partial x \partial y} \tag{10}$$

It is found from equations (1) by eliminating u and v that:

$$\begin{aligned} \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - \frac{\partial^2 e_{xy}}{\partial x \partial y} &= \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \\ &- \left(\frac{\partial^2 w_o}{\partial x \partial y}\right)^2 + \frac{\partial^2 w_o}{\partial x^2} \frac{\partial^2 w_o}{\partial y^2} - \frac{1}{r} \frac{\partial^2 w}{\partial x^2} + \frac{1}{r} \frac{\partial^2 w_o}{\partial x^2} \end{aligned} \tag{11}$$

By introducing (10) in (8) and substituting the results in (11), the following differential equation for F is obtained:

$$A\frac{\partial^4 F}{\partial x^4}+B\frac{\partial^4 F}{\partial y^4}+C\frac{\partial^4 F}{\partial x^2\partial y^2}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w_o}{\partial x\partial y}\right)^2$$

$$+\frac{\partial^2 w_o}{\partial x^2}\frac{\partial^2 w_o}{\partial y^2}-\frac{1}{r}\frac{\partial^2 w}{\partial x^2}+\frac{1}{r}\frac{\partial^2 w_o}{\partial x^2} \tag{12}$$

where:

$$A=\frac{1}{E_b},\quad B=\frac{1}{E_a},\quad C=\frac{1}{\mu_m}-\frac{2\sigma_{xy}}{E_a} \tag{13}$$

It is readily established that the following expression represents the energy of extensional deformation of a rectangular portion of the shell with edges of length a and b:

$$W_1=\frac{h}{2}\int_0^a\int_0^b\left[B\overline{X}_x^{\,2}+A\overline{Y}_y^{\,2}-\frac{2\sigma_{xy}}{E_a}\overline{X}_x\,\overline{Y}_y+\frac{1}{\mu_m}\overline{X}_y^{\,2}\right]dy\,dx \tag{14}$$

Form of Buckles and Initial Irregularities

The stress components $\overline{X}_x$, $\overline{Y}_y$, and $\overline{X}_y$ in (14) are derived from a stress function F, satisfying the differential equation (12), which involves derivatives of w_o and w representing the initial and deformed middle surface of the shell. For w, the inward radial deflection, the following form will be chosen:

$$\frac{w}{r}=g+\delta\cos^2(\beta y-\alpha x)\cos^2(\beta y+\alpha x) \tag{15}$$

where

$$\beta=\frac{\pi}{b},\quad \alpha=\frac{\pi}{a} \tag{16}$$

The nodal lines of the trigonometric portion of equation (15) are shown in figure 2. The displacement w is positive inward. In equation (16), a and b represent the length and width, respectively, of a diamond. The initial irregularities will be assumed to have the form (15). This is done for the purpose of simplifying the calculations. Then w_o is chosen in the following form:

$$\frac{w_o}{r} = g_o + \delta_o \cos^2 (\beta y - \alpha x) \cos^2 (\beta y + \alpha x) \tag{17}$$

An initial flat spot on the surface of the cylinder could be described roughly by equation (17). An initial irregularity was introduced here, as it was in report No. 1322-A (7), to obtain a qualitative description of its influence in causing an isolated buckle to develop in its vicinity. If the initial depth of an irregularity of the form (17) is very small, the dimensions of the area that it occupies are not very important. For this reason in order to simplify the calculations, the dimensions a and b in equation (17) are taken to be the same as those in equation (15). From the qualitative description that is obtained of the development of an isolated buckle, conclusions were drawn in report No. 1322-A that led to the derivation of the final formulas from the analysis for the case $w_o = 0$.

The details of substituting (15) and (17) in (12), of obtaining the stress function F and the stress components $\overline{X}_x$, $\overline{Y}_y$, and $\overline{X}_y$, of substituting these stress components in equation (14), and of related operations are identical with the corresponding operations performed in report No. 1322-A (7). Reference is therefore made to equations (21) and (31) of that report. The following differences in notation should be noted:

Notation of Report No. 1322-A	Notation of Present Report
H	$E_a E_b$
$\frac{E_L \sigma_{TL}}{H}$	$\frac{\sigma_{yx}}{E_b} = \frac{\sigma_{xy}}{E_a}$
$f,\ f_o$	$\delta,\ \delta_o$
$X',\ Y',\ X_y'$	$\overline{X},\ \overline{Y},\ \overline{X}_y$

From equation (14), the energy of extensional deformation W_1 is then found to be:

$$W_1 = \frac{hab}{8} \left\{ 4r^4 \alpha^4 \beta^4 (\delta^2 - \delta_o^2)^2 \left[\frac{1}{128 B \beta^4} + \frac{\left(1 - \frac{1}{r^2\beta^2 (\delta + \delta_o)}\right)^2}{128 A \alpha^4} \right.\right.$$

$$+ \frac{1}{16 (A\alpha^4 + 81 B\beta^4 + 9C\alpha^2\beta^2)} + \frac{1}{16(81A\alpha^4 + B\beta^4 + 9C\alpha^2\beta^2)}$$

$$\left.\left. + \frac{4\left(2 - \frac{1}{r^2\beta^2 (\delta + \delta_o)}\right)^2 + 1}{64(A\alpha^4 + B\beta^4 + C\alpha^2\beta^2)} \right] + 4Bp^2 + 4Ac_1^2 + \frac{8\sigma_{xy}}{E_a} c_1 p \right\} \tag{18}$$

This equation is equation (31) of report No. 1322-A (7) with the proper values inserted for the former abbreviations M and S. The quantities p and c_1 represent the mean compressive stress and the mean circumferential stress, respectively.

If n is the number of buckles in a circumference, the width b of an individual buckle and n are related by the equation:

$$b = \frac{2\pi r}{n} \tag{19}$$

Then:

$$\beta = \frac{\pi}{b} = \frac{n}{2r} \tag{20}$$

It will be convenient to denote the ratio $\frac{b}{a}$ of the dimensions of a buckle by:

$$z = \frac{b}{a} = \frac{\alpha}{\beta} \tag{21}$$

In the expression obtained from (18) by using equations (20) and (21), let:

$$\eta = n^2 \frac{h}{r}, \quad \xi = \delta \frac{r}{h}, \quad \xi_o = \delta_o \frac{r}{h} \tag{22}$$

$$K_1 = Az^4 + 81B + 9Cz^2 \tag{23}$$

$$K_2 = 81Az^4 + B + 9Cz^2 \tag{24}$$

$$K_3 = Az^4 + B + Cz^2 \tag{25}$$

$$e_1 = \frac{z^4}{4096B} + \frac{1}{4096A} + \frac{z^4}{512K_1} + \frac{z^4}{512K_2} + \frac{17z^4}{2048K_3} \tag{26}$$

$$e_2 = \frac{1}{512A} + \frac{z^4}{32K_3} \tag{27}$$

$$e_3 = \frac{1}{256A} + \frac{z^4}{32K_3} \tag{28}$$

Equation (18) then becomes:

$$W_1 = hab \left\{ \left[e_1 \eta^2 (\xi^2 - \xi_o^2)^2 - e_2 \eta (\xi^2 - \xi_o^2)(\xi - \xi_o) + e_3 (\xi - \xi_o)^2 \right] \frac{h^2}{r^2} + \frac{Bp^2}{2} + \frac{Ac_1^2}{2} + \frac{\sigma_{xy}}{E_a} c_1 p \right\} \tag{29}$$

Flexu:.l Energy of the Shell

To determine flexural energy of the shell, the following simplified expressions for the changes in curvature and unit twist are used:

$$\frac{\partial^2 (w - w_o)}{\partial x^2}, \quad \frac{\partial^2 (w - w_o)}{\partial y^2}, \quad \frac{\partial^2 (w - w_o)}{\partial x \partial y} \tag{30}$$

A discussion of the approximations involved will be found in a paper by Donnell (1). These expressions were used by von Kàrman and Tsien (16) and by March (7). The expressions (30) are exactly those used in calculating the flexural energy of a flat sandwich plate. The approximate flexural energy of such a plate was found by March (9) and by Ericksen and March (2) by using the "tilting" method of Williams, Leggett, and Hopkins (4, 18).

In this method it is assumed that any line in the core that is initially straight and normal to the undeformed plate will remain straight after the deformation, but will deviate in the x and y directions from the normal to the deformed plate by amounts that are expressed by the parameters k and k'. These parameters are determined by an energy method. These "tilting" factors k and k' are introduced as well as two quantities q and q' that determine the positions of the surfaces in which, respectively, the components u and v of the displacement in the core vanish. The letters k' and q' replace h and r, respectively, of report No. 1583-B, because h and r have already been used in the present report. The following derivation of the expression for the flexural energy follows closely that used for the flat sandwich panel in Forest Products Laboratory Report No. 1583-B (2), to which reference is made for further details. For the sake of simplicity in writing, the initial irregularity w_o will be for the present taken equal to zero. It will then be introduced in the final steps by replacing w by $w - w_o$.

The components of displacement in the core (fig. 3) are taken to be:

$$\begin{aligned} u_c &= - k (\zeta - q) \frac{\partial w}{\partial x} \\ v_c &= - k' (\zeta - q') \frac{\partial w}{\partial y} \\ w_c &= w (x, y) \end{aligned} \tag{31}$$

Thus $\zeta = q$ denotes the surface in which the components of displacement in the $\underline{x}$ direction vanish and $\underline{k}$ is the parameter describing the inclination in the $\underline{x}$ direction of the respective plane sections to the normal to the deformed surface. Similarly $\underline{q'}$ and $\underline{k'}$ are related to the displacements in the $\underline{y}$ direction. These four quantities are to be determined in such a way that the flexural energy associated with a prescribed deflection $\underline{w}$ is a minimum.

To arrive at expressions for the components of displacement in the facings, it is noted that the continuity of the displacement at the facing-to-core bonds requires that the components (31), evaluated at $\zeta = 0$ and $\zeta = c$, shall be those at the inner surfaces of the facings f_1 and f_2, respectively. Within each facing, the components of displacement are assumed to be such that a straight line initially normal to the undeformed surface of the plate will be straight and normal to the deformed surface. Accordingly, the components of displacement in the facings $\underline{f_1}$ and $\underline{f_2}$, respectively, are:

$$u_1 = (kq - \zeta)\frac{\partial w}{\partial x}$$

$$v_1 = (k'q' - \zeta)\frac{\partial w}{\partial y} \qquad (32)$$

$$w_1 = w\,(x,\; y)$$

and

$$u_2 = -\left[k\,(c - q) + \zeta - c\right]\frac{\partial w}{\partial x}$$

$$v_2 = -\left[k'\,(c - q') + \zeta - c\right]\frac{\partial w}{\partial y} \qquad (33)$$

$$w_2 = w\,(x,\; y)$$

The components of strain in the core $\underline{c}$ and facings $\underline{f_1}$ and $\underline{f_2}$ will be denoted by the superscripts $\underline{c}$, 1, and 2, respectively.

From (31), the transverse shear strains in the core are:

$$e_{\zeta x}^{(c)} = (1 - k)\frac{\partial w}{\partial x} \qquad e_{y\zeta}^{(c)} = (1 - k')\frac{\partial w}{\partial y} \qquad (34)$$

The effect of the remaining strains in the core is assumed to be negligible.

In finding the strain energy of the facings in the bending of the plate (or shell), it is convenient to consider the components of strain in the facings to result from the superposition of two states of strain. The first of these consists of the membrane strains in the facings associated with flexure, that is the strain in their middle surfaces. From (32) and (33), these strains are found to be:

$$e_{xx}^{(1)} = (kq + \frac{f_1}{2}) \frac{\partial^2 w}{\partial x^2}$$

$$e_{yy}^{(1)} = (k'q' + \frac{f_1}{2}) \frac{\partial^2 w}{\partial y^2}$$

$$e_{xy}^{(1)} = (kq + k'q' + f_1) \frac{\partial^2 w}{\partial x \partial y} \tag{35}$$

and

$$e_{xx}^{(2)} = -\left[k\,(c - q) + \frac{f_2}{2}\right] \frac{\partial^2 w}{\partial x^2}$$

$$e_{yy}^{(2)} = -\left[k'\,(c - q') + \frac{f_2}{2}\right] \frac{\partial^2 w}{\partial y^2}$$

$$e_{xy}^{(2)} = -\left[k\,(c - q) + k'\,(c - q') + f_2\right] \frac{\partial^2 w}{\partial x \partial y} \tag{36}$$

The second state of strain in the facings is that associated with their bending about their own middle surfaces. This state, in either facing, has the components:

$$e'_{xx} = -\zeta' \frac{\partial^2 w}{\partial x^2}, \quad e'_{yy} = -\zeta' \frac{\partial^2 w}{\partial y^2}, \quad e'_{xy} = -2\zeta' \frac{\partial^2 w}{\partial x \partial y} \tag{37}$$

where ζ' is measured from the middle surface of the facing under consideration.

The strain energy in the core or facings is given by the expression (6, 8).

$$U = \frac{1}{2\lambda} \iiint \Big[E_x\, e^2{}_{xx} + E_y\, e^2{}_{yy} + 2E_x\, \sigma_{yx}\, e_{xx} e_{yy}$$

$$+ \lambda\mu_{xy}\, e^2{}_{xy} + \lambda\mu_{y\zeta}\, e^2{}_{y\zeta} + \lambda\mu_{\zeta x}\, e^2{}_{\zeta x}\Big]\, d\zeta\, dy\, dx \tag{38}$$

where for the material under consideration (core or facing), $\lambda = 1 - \sigma_{xy}\sigma_{yx}$; E_x and E_y are Young's moduli; μ_{xy}, $\mu_{y\zeta}$, and $\mu_{\zeta x}$ are moduli of rigidity; and σ_{xy} and σ_{yx} are Poisson's ratios. Primed letters will denote the elastic constants of the core material and unprimed letters will denote those of the facing material. The integration indicated in formula (38) is to be carried out over the area OABC of figure 2 and the thickness of the core or facings.

The energy in the core is obtained by substituting expressions (34) into (38), the remaining strains in the latter formula being neglected as previously stated. After integrating with respect to ζ over the thickness of the core, the expression for the energy, denoted by U_c, is

$$U_c = \frac{c}{2} \int_0^a \int_0^b \left[\mu'_{\zeta x} \; (1 - k)^2 \left(\frac{\partial w}{\partial x}\right)^2 + \mu'_{y\zeta} \; (1 - k')^2 \left(\frac{\partial w}{\partial y}\right)^2 \right] dy \; dx \tag{39}$$

The strain energy in the facings associated with the membrane strains is the sum of the energies obtained from (35) and (36). With the substitution of these expressions into (38) one obtains, after integration with respect to ζ, the following expression which is denoted by U_M.

$$U_M = \frac{1}{2\lambda} \int_0^a \int_0^b \left[E_x \left\{ f_1 \; (kq + \frac{f_1}{2})^2 + f_2 \left(k \; (c - q) + \frac{f_2}{2}\right)^2 \right\} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right.$$

$$+ E_y \left\{ f_1 \; (k'q' + \frac{f_1}{2})^2 + f_2 \left(k' \; (c - q') + \frac{f_2}{2}\right)^2 \right\} \left(\frac{\partial^2 w}{\partial y^2}\right)^2$$

$$+ 2E_x\sigma_{yx} \left\{ f_1 \; (kq + \frac{f_1}{2})(k'q' + \frac{f_1}{2}) + f_2 \left(k \; (c - q) + \frac{f_2}{2}\right)\left(k' \; (c - q') + \frac{f_2}{2}\right) \right\} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$$

$$\left. + \lambda\mu_{xy} \left\{ f_1 \; (kq + k'q' + f_1)^2 + f_2 \left(k \; (c - q) + k' \; (c - q') + f_2\right)^2 \right\} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dy \; dx \tag{40}$$

The strain energy in the facings associated with the flexural strain, U_F, is obtained by substituting expressions (37) into (38) and integrating over the volume of each facing. After integrating with respect to ζ,

$$U_F = \left(\frac{f_1^{\,3} + f_2^{\,3}}{24\lambda}\right) \int_0^a \int_0^b \left[E_x \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + E_y \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2E_x\sigma_{yx} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right.$$

$$\left. + \lambda\mu_{xy} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dy \; dx \tag{41}$$

Now in all of the expressions (39), (40), and (41) replace w by $w - w_o$. The flexural energy W_2 of the region OABC of the shell is the sum of U_M, U_F, and U_c.

For equilibrium, the "tilting" factors k and k' and the ordinates q and q' of the neutral surfaces are to be chosen so that the total energy is a minimum. But these factors appear only in the flexural energy W_2. Hence, they must be chosen to satisfy the conditions

$$\frac{\partial W_2}{\partial (kq)} = 0, \quad \frac{\partial W_2}{\partial (k'q')} = 0, \quad \frac{\partial W_2}{\partial k} = 0, \quad \frac{\partial W_2}{\partial k'} = 0$$

By proceeding exactly as in report No. 1583-B (2), the quadratic form (A14) of that report with k' and q' replacing h and r, respectively, is obtained for W_2. The coefficients B_i in equation (A14) are defined by equations (A15) in terms of the quantities A_i, which are defined by:

$$A_1 = \frac{1}{\lambda} \int_0^a \int_0^b \left[E_x \left(\frac{\partial^2 (w - w_o)}{\partial x^2} \right)^2 + \lambda \mu_{xy} \left(\frac{\partial^2 (w - w_o)}{\partial x \partial y} \right)^2 \right] dy\, dx \tag{42}$$

$$A_2 = \frac{1}{\lambda} \int_0^a \int_0^b \left[E_x \sigma_{yx} \frac{\partial^2 (w - w_o)}{\partial x^2} \frac{\partial^2 (w - w_o)}{\partial y^2} + \lambda \mu_{xy} \left(\frac{\partial^2 (w - w_o)}{\partial x \partial y} \right)^2 \right] dy\, dx \tag{43}$$

$$A_3 = \frac{1}{\lambda} \int_0^a \int_0^b E_y \left(\frac{\partial^2 (w - w_o)}{\partial y^2} \right)^2 + \lambda \mu_{xy} \left(\frac{\partial^2 (w - w_o)}{\partial x \partial y} \right)^2 dy\, dx \tag{44}$$

$$A_4 = \int_0^a \int_0^b \mu'_{\zeta x} \left(\frac{\partial (w - w_o)}{\partial x} \right)^2 dy\, dx \tag{45}$$

$$A_5 = \int_0^a \int_0^b \mu'_{y\zeta} \left(\frac{\partial (w - w_o)}{\partial y} \right)^2 dy\, dx \tag{46}$$

On factoring out the common factor, it is found that:

$$2W_2 = r^2 (\delta - \delta_o)^2 ab \Big[B_1' (kq)^2 + 2B_2' (kq)(k'q') + B_3' (k'q')^2$$

$$+ 2B_4' (kq) k + 2B_5' (kq) k' + 2B_5' (k'q') k + 2B_6' (k'q') k'$$

$$+ B_7' k^2 + 2B_8' kk' + B_9' k'^2 + 2B_{10}' (kq) + 2B_{11}' (k'q')$$

$$+ 2B_{12}' k + 2B_{13}' k' + B_{14}' + B_{15}' \Big] \tag{47}$$

where the quantities $\underline{B_i'}$ are defined in terms of the quantities $\underline{A_i'}$ by equations (A15) of report No. 1583-B, each $\underline{A_i'}$ replacing the corresponding $\underline{A_i}$ in those equations. The quantity in brackets in equation (47) corresponds to $\underline{2U'}$ in report No. 1583-B. (Note that equation (A22) of that report should read $P = 2U'$).

It is easy to see that the steps of imposing the conditions

$$\frac{\partial W_2}{\partial (kq)} = 0, \quad \frac{\partial W_2}{\partial (k'q')} = 0, \quad \frac{\partial W_2}{\partial k} = 0, \quad \frac{\partial W_2}{\partial k'} = 0$$

and of determining $\underline{kq}$, $\underline{k'q'}$, $\underline{k}$, and $\underline{k'}$ and substituting their values in the expression (47) for $\underline{2W_2}$ are identical with those taken in report No. 1583-B (2) and that $\underline{2W_2}$ is equal to the right-hand member of equation (A25) of that report multiplied by $r^2 (\delta - \delta_o)^2$ ab. It is concluded from equations (A26), (A27), and (A28) of report No. 1583-B that:

$$W_2 = 1/2\, r^2 (\delta - \delta_o)^2 ab \left\{ \frac{I \left[A_1' + 2A_2' + A_3' + (A_1'A_3' - A_2'^2)\left(\frac{\phi}{A_4'} + \frac{\phi}{A_5'}\right) \right]}{1 + \frac{A_1'\phi}{A_4'} + \frac{A_3'\phi}{A_5'} + \frac{\phi^2 (A_1'A_3' - A_2'^2)}{A_4'A_5'}} \right.$$

$$\left. + I_f (A_1' + 2A_2' + A_3') \right\} \tag{48}$$

where $\underline{I}$, $\underline{I_f}$, and $\underline{\phi}$ are defined by:

$$I = \frac{f_1 f_2}{f_1 + f_2} \left(c + \frac{f_1 + f_2}{2} \right)^2 \tag{49}$$

$$I_f = \frac{f_1^3 + f_2^3}{12} \tag{50}$$

$$\phi = \frac{c\, f_1\, f_2}{f_1 + f_2} \tag{51}$$

and $\underline{A_i'}$ by:

$$A_1' = \frac{1}{\lambda} \left(3\, E_x \alpha^4 + \lambda \mu_{xy} \alpha^2 \beta^2\right) \tag{52}$$

$$A_2' = \frac{1}{\lambda} \left(E_x \sigma_{yx} + \lambda \mu_{xy}\right) \alpha^2 \beta^2 \tag{53}$$

$$A_3' = \frac{1}{\lambda} \left(3\, E_y \beta^4 + \lambda \mu_{xy} \alpha^2 \beta^2\right) \tag{54}$$

$$A_4' = \frac{3\alpha^2\, \mu'_{\zeta x}}{8}, \qquad A_5' = \frac{3\beta^2\, \mu'_{y\zeta}}{8} \tag{55}$$

Note that

$$A_1' + 2A_2' + A_3' = \frac{\beta^4}{\lambda} K_4$$

where

$$K_4 = 3E_x z^4 + 3E_q + 2(E_x \sigma_{yx} + 2\lambda \mu_{xy})\, z^2$$

and introduce the following abbreviations in the expressions for $\underline{A_1'}$, $\underline{A_2'}$, and $\underline{A_3'}$:

$$A_1' = \frac{\beta^4}{\lambda} d_1, \quad A_2' = \frac{\beta^4}{\lambda} d_2, \quad A_3' = \frac{\beta^4}{\lambda} d_3 \tag{56}$$

where:

$$d_1 = 3E_x z^4 + \lambda\mu_{xy} z^2$$

$$d_2 = (E_x \sigma_{yx} + \lambda\mu_{xy}) z^2 \tag{57}$$

$$d_3 = 3E_y + \lambda\mu_{xy} z^2$$

Also,

$$A'_4 = \frac{3\alpha^2 \mu'_{\zeta x}}{8} = \frac{3\beta^2 \mu'_{\zeta x} z^2}{8} \tag{58}$$

Substituting these expressions for A'_i equation (48) becomes

$$W_2 = \frac{1}{2\lambda} r^2 (\delta - \delta_o)^2 ab\beta^4 \left\{ \frac{I\left[K_4 + \frac{8\beta^2\phi}{3\lambda}(d_1 d_3 - d_2^2)\left(\frac{1}{\mu'_{\zeta x} z^2} + \frac{1}{\mu'_{y\zeta}}\right)\right]}{1 + \frac{8\beta^2 d_1 \phi}{3\lambda\mu'_{\zeta x} z^2} + \frac{8\beta^2 d_3 \phi}{3\lambda\mu'_{y\zeta}} + \frac{64\beta^4\phi^2 (d_1 d_3 - d_2^2)}{9\lambda^2 \mu'_{\zeta x} \mu'_{y\zeta} z^2}} + I_f K_4 \right\} \tag{59}$$

The following transformations are made by using equations (20) and (22):

$$\frac{8\beta^2\phi}{3\lambda\mu'_{\zeta x}} = \frac{2n^2\phi}{3\lambda\mu'_{\zeta x} r^2} = \frac{2\eta\phi}{3\lambda\mu'_{\zeta x} rh} = \frac{\eta}{E_x} S_x$$

where

$$S_x = \frac{2E_x\phi}{3\lambda\mu'_{\zeta x} rh} \tag{60}$$

and

$$\frac{8\beta^2\phi}{3\lambda\mu'_{y\zeta}} = \frac{\eta}{E_x} S_y$$

where

$$S_y = \frac{2E_x\phi}{3\lambda\mu'_{y\zeta} rh} \tag{61}$$

The coefficient of the expression in brackets in equation (59) is also transformed by using equations (20) and (22). The expression for W_2 becomes:

$$W_2 = \frac{(\xi - \xi_o)^2 \, ab \, \eta^2 h^3}{32\lambda r^2} \left\{ \frac{(I/h^3) \left[K_4 + (d_1 d_3 - d_2{}^2) \frac{\eta}{E_x} \left(\frac{S_x}{z^2} + S_y\right)\right]}{1 + \frac{\eta d_1 S_x}{E_x z^2} + \frac{\eta d_3 S_y}{E_x} + \frac{\eta^2 (d_1 d_3 - d_2{}^2) S_x S_y}{E_x{}^2 z^2}} + (I_f/h^3) K_4 \right\} \tag{62}$$

Or

$$W_2 = e_4 \, ab \, \eta^2 \, (\xi - \xi_o)^2 \, \frac{h^3}{r^2} \tag{63}$$

where

$$e_4 = \frac{1}{32\lambda} \left\{ \frac{(I/h^3) \left[K_4 + (d_1 d_3 - d_2{}^2) \frac{\eta}{E_x} \left(\frac{S_x}{z^2} + S_y\right)\right]}{1 + \frac{\eta d_1 S_x}{E_x z^2} + \frac{\eta d_3 S_y}{E_x} + \frac{\eta^2 (d_1 d_3 - d_2{}^2) S_x S_y}{E_x{}^2 z^2}} + (I_f/h^3) K_4 \right\} \tag{64}$$

Virtual Work of the Compressive Load

Exactly as in equation (35) of report No. 1322A (7), the virtual work, W_3, of the compressive load, calculated for the region OABC, figure 2, is found in the notation of the present report to be:

$$\begin{aligned} W_3 &= abh \left[Bp^2 + \frac{\sigma_{xy}}{E_a} pc_1 + \frac{3}{16} r^2 \alpha^2 (\delta^2 - \delta_o{}^2) p \right] \\ &= abh \left[Bp^2 + \frac{\sigma_{xy}}{E_a} pc_1 + e_5 \, \eta \, (\xi^2 - \xi_o{}^2) \, p \, \frac{h}{r} \right] \end{aligned} \tag{65}$$

where

$$e_5 = \frac{3}{64} z^2 \tag{66}$$

It will be convenient to consider the mean energy per unit volume of the cylindrical shell. Hence:

$$W = (W_1 + W_2 - W_3)/abh \tag{67}$$

In accordance with equations (29), (63), and (65):

$$W = \left[e_1\eta^2(\xi^2 - \xi_o^2)^2 - e_2\eta(\xi^2 - \xi_o^2)(\xi - \xi_o) + e_3(\xi - \xi_o)^2 + e_4\eta^2(\xi - \xi_o)^2\right]\frac{h^2}{r^2} - e_5\eta(\xi^2 - \xi_o^2)\,p\,\frac{h}{r} - \frac{Bp^2}{2} + \frac{Ac_1^2}{2} \tag{68}$$

The Buckling Stress

For equilibrium, the derivatives of W with respect to the various parameters g, c_1, ξ, η, and z vanish. From the condition

$$\frac{\partial W}{\partial c_1} = 0, \text{ it follows that } c_1 = 0 \tag{69}$$

Now c_1 denotes the mean circumferential stress. The parameter g appears only in the expression for c_1 as given by equation (30) of report No. 1322-A (7). The fact that c_1 vanishes implies that g, which describes a uniform radial expansion of the cylinder, takes on such a value that the mean circumferential stress vanishes. Further consideration of the parameter g is not necessary.

From the condition $\frac{\partial W}{\partial \xi} = 0$, it follows that:

$$p = \left[2e_1\eta(\xi + \xi_o) - \frac{e_2(3\xi + \xi_o)}{2\xi} + \frac{e_3}{\eta\xi} + \frac{e_4\eta}{\xi}\right]\frac{(\xi - \xi_o)}{e_5}\,\frac{h}{r} \tag{70}$$

where p, as previously noted, is the mean compressive stress.

Let:

$$\gamma_1 = \frac{e_1}{E_a}, \quad \gamma_2 = \frac{e_2}{E_a}, \quad \gamma_3 = \frac{e_3}{E_a}, \quad \gamma_4 = \frac{e_4}{E_a} \tag{71}$$

Then (70) can be written:

$$p = E_a\left[2\gamma_1\eta(\xi + \xi_o) - \frac{\gamma_2(3\xi + \xi_o)}{2\xi} + \frac{\gamma_3}{\eta\xi} + \frac{\gamma_4\eta}{\xi}\right]\frac{(\xi - \xi_o)}{e_5}\,\frac{h}{r} \tag{72}$$

The mean compressive strain ϵ is expressed by:

$$-\epsilon = \frac{1}{ab}\int_{o}^{b} dy \left[\frac{1}{a}\int_{o}^{a} \frac{\partial u}{\partial x}\, dx\right]$$

As in report No. 1322-A (7), it is found that:

$$\epsilon = Bp + \frac{3\eta}{64} z^2 (\xi^2 - \xi_o^2) \frac{h}{r} \tag{73}$$

The further equilibrium conditions

$$\frac{\partial W}{\partial \eta} = 0 \text{ and } \frac{\partial W}{\partial z} = 0$$

are to be satisfied. The first of these is associated with the number of buckles in a circumference or with the width of an individual buckle, and the second with the ratio b/a of the width of a buckle to its length. It is not analytically feasible to use these conditions in connection with equation (68).

The following method of arriving at the critical value of p is based upon an extended discussion in report No. 1322-A (7). Briefly, it was considered that an isolated initial irregularity would increase in size and depth with increasing mean compressive stress p. It was therefore considered that the load-mean compressive strain curve, with p as a function of ϵ, for a given small initial depth of irregularity would be the envelope of the family of curves for p as a function of ϵ, drawn for a series of values of η by combining equations (71) and (72). On taking into consideration the possibility of jumps from one energy level to another, it was concluded that the critical values of p would scatter considerably, as they actually do in test, depending upon the depth of the initial irregularity and the characteristics of the loading process. It was noted that the value of p at the relative minimum point on the envelope of the curves for p as a function of ϵ, drawn for $\xi_o = 0$, was intermediate among the possible critical values of p. This minimum was accordingly chosen as the "theoretical" critical stress, because it could be conveniently determined by finding a relative minimum of p as a function of ξ and η. It is necessary to employ numerical methods to determine the relative minimum value of p.

In report No. 1322-A (7), the aspect ratio z of the buckles was assumed on the basis of experimental observations before the minimization of p was undertaken. Here, because of the influence of shear deformation in the core, a suitable value to assign to z can not be estimated.

Equation (71), with $\xi_o = 0$, can be written in the form:

$$p = KE_a \frac{h}{r} \tag{74}$$

where

$$K = \frac{32}{3z^2}\left[4\gamma_1\ \eta\xi^2 - 3\gamma_2\xi + \frac{2\gamma_3}{\eta} + 2\gamma_4\ \eta\right] \tag{75}$$

The mean compressive stress, p_f, in the facings is related to the mean compressive stress in the shell by the equation

$$p = \frac{p_f\ (f_1 + f_2)}{h} \tag{76}$$

On recalling the definition of E_a, it is seen that equation (74) can be written

$$p_f = KE_x\ \frac{h}{r} \tag{77}$$

For a relative minimum of p_f, the condition

$\dfrac{\partial K}{\partial \xi} = 0$ must be satisfied.

From this condition, it follows that:

$$\xi = \frac{3\gamma_2}{8\gamma_1\ \eta} \tag{78}$$

The substitution of this value of ξ in (75) yields

$$K = \frac{64}{3z^2}\left(\frac{\gamma_3}{\eta} - \frac{9\gamma_2^{\,2}}{32\gamma_1\eta} + \gamma_4\ \eta\right) \tag{79}$$

In equation (79), K is a function of n and z, which occur in the definitions of η, γ_1, γ_2, γ_3, and γ_4. By using the definitions of the quantities E_a, E_b, and μ_m that appear through the symbols A, B, and C in the equations (26), (27), and (28), the following expressions are obtained for γ_1, γ_2, and γ_3 (see equations (71), (26), (27), and (28):

$$\gamma_1 = \frac{z^4}{4096} + \frac{E_y}{4096\ E_x} + \frac{z^4}{512\ \left(z^4\dfrac{E_x}{E_y} + 81 + \dfrac{9z^2E_x}{\mu_{xy}} - 18\sigma_{xy}z^2\right)}$$

$$+ \frac{z^4}{512\ \left(81z^4\dfrac{E_x}{E_y} + 1 + 9z^2\dfrac{E_x}{E_y} - 18\sigma_{xy}z^2\right)} + \frac{17z^4}{2048\ \left(z^4\dfrac{E_x}{E_y} + 1 + \dfrac{z^2E_x}{\mu_{xy}} - 2\sigma_{xy}z^2\right)} \tag{80}$$

$$\gamma_2 = \frac{E_y}{512E_x} + \frac{z^4}{32\left(z^4 \frac{E_x}{E_y} + 1 + \frac{z^2 E_x}{\mu_{xy}} - 2\sigma_{xy} z^2\right)} \tag{81}$$

$$\gamma_3 = \frac{E_y}{256E_x} + \frac{z^4}{32\left(z^4 \frac{E_x}{E_y} + 1 + \frac{z^2 E_x}{\mu_{xy}} - 2\sigma_{xy} z^2\right)} \tag{82}$$

In obtaining the expression for γ_4 from equations (63) and (70), it is convenient to introduce the notation

$$T = 3z^4 + 3\frac{E_y}{E_x} + \frac{2}{E_x}(E_x\sigma_{yx} + 2\lambda\mu_{xy})\, z^2 \tag{83}$$

so that

$$K_4 = E_x\, T \tag{84}$$

and

$$\frac{K_4}{E_a} = \frac{hT}{f_1 + f_2} \tag{85}$$

then

$$\gamma_4 = \frac{T}{32\lambda h^2 (f_1 + f_2)} \left[\frac{I + \frac{I}{K_4}(d_1 d_3 - d_2^2)\frac{\eta}{E_x}\left(\frac{S_x}{z^2} + S_y\right)}{1 + \frac{\eta d_1 S_x}{E_x z^2} + \frac{\eta d_3 S_y}{E_x} + \frac{\eta^2 (d_1 d_3 - d_2^2) S_x S_y}{E_x^2 z^2}} + I_f \right] \tag{86}$$

Buckling Stress of Sandwich Constructions with Isotropic Facings and Orthotropic or Isotropic Core

For isotropic facings, considerable simplifications can be made. In this case

$$E_x = E_y = (E_x\sigma_{yx} + 2\lambda\mu_{xy}) = E, \quad \mu_{xy} = \mu = E/2\,(1 + \sigma)$$

$$\sigma_{xy} = \sigma_{yx} = \sigma$$

Then

$$\gamma_1 = \frac{1 + z^4}{4096} + \frac{z^4}{512\,(z^2 + 9)^2} + \frac{z^4}{512\,(9z^2 + 1)^2} + \frac{17\,z^4}{2048\,(1 + z^2)^2} \tag{87}$$

$$\gamma_2 = \frac{1}{512} + \frac{z^4}{32\,(1 + z^2)^2} \tag{88}$$

$$\gamma_3 = \frac{1}{256} + \frac{z^4}{32\,(1 + z^2)^2} \tag{89}$$

$$\gamma_4 = \frac{T}{32\lambda\, h^2\,(f_1 + f_2)} \left[\frac{I + \dfrac{I\,(d_1 d_3 - d_2^2)\,\eta}{(3z^4 + 3 + 2z^2)\,E^2} \left(\dfrac{S_x}{z^2} + S_y\right)}{1 + \dfrac{\eta d_1 S_x}{E z^2} + \dfrac{\eta d_3 S_y}{E} + \dfrac{\eta^2 S_x S_y\,(d_1 d_3 - d_2^2)}{E^2\,z^2}} + I_f \right] \tag{90}$$

where

$$T = 3z^4 + 3 + 2z^2 \tag{91}$$

$$S_x = \frac{2\,\phi\,E}{3\lambda\,\mu'_{\zeta x}\,rh} \qquad S_y = \frac{2\,\phi\,E}{3\lambda\,\mu'_{y\zeta}\,rh} \tag{92}$$

After some manipulation involving substitution of expressions for $\underline{\gamma_1}$, $\underline{\gamma_2}$, $\underline{\gamma_3}$, and $\underline{\gamma_4}$, formula (79) for $\underline{K}$ for sandwich construction with isotropic facings and orthotropic core can be written as:

$$K = \frac{M_1}{\eta} + \frac{2I}{3\lambda\, h^2\,(f_1 + f_2)} \left[\frac{M_2\eta + M_3\eta^2 S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x^{\,2}} + \frac{I_f}{I}\; M_2\eta \right] \tag{93}$$

where

$$M_1 = \frac{64}{3z^2}\left(\gamma_3 - \frac{9\gamma_2^{\,2}}{32\gamma_1}\right) \tag{94}$$

$$M_2 = \frac{T}{z^2} \tag{95}$$

$$M_3 = \frac{(d_1 d_3 - d_2^{\,2})}{E^2\,z^2}\left(\frac{1}{z^2} + \theta\right) \tag{96}$$

$$M_4 = \left(\frac{d_1}{z^2} + d_3\theta\right)\frac{1}{E} \tag{97}$$

$$M_5 = \frac{(d_1 d_3 - d_2^{\,2})\,\theta}{E^2\, z^2} \tag{98}$$

$$\theta = \frac{S_y}{S_x} \quad \text{or} \quad \theta = \frac{\mu'_{\zeta x}}{\mu'_{y\zeta}} \tag{99}$$

$$d_1 = 3Ez^4 + \frac{1}{2}(1 - \sigma)\, Ez^2 \tag{100}$$

$$d_2 = E\sigma z^2 + \frac{1}{2}(1 - \sigma)\, Ez^2 \tag{101}$$

$$d_3 = 3E + \frac{1}{2}(1 - \sigma)\, Ez^2 \tag{102}$$

If σ is taken to be $\frac{1}{4}$, then:

$$d_1 = 3Ez^2\left(z^2 + \frac{1}{8}\right) \tag{103}$$

$$d_2 = \frac{5}{8}\, Ez^2 \tag{104}$$

$$d_3 = 3E\left(\frac{z^2}{8} + 1\right) \tag{105}$$

If also $\underline{\gamma_1}$, $\underline{\gamma_2}$, $\underline{\gamma_3}$, and $\underline{T}$ are expressed in terms of z and θ (Eq. 87, 88, 89, and 90) then the following expressions can be used in formula 89:

$$M_1 = \frac{1}{12z^2} + \frac{2z^2}{3(1 + z^2)^2} - \frac{3\left[\frac{1}{64z} + \frac{z^3}{4(1 + z^2)^2}\right]^2}{\frac{1 + z^4}{128} + \frac{z^4}{16(z^2 + 9)^2} + \frac{z^4}{16(9z^2 + 1)^2} + \frac{17z^4}{64(1 + z^2)^2}} \tag{106}$$

$$M_2 = 3z^2 + 2 + \frac{3}{z^2} \tag{107}$$

$$M_3 = \frac{1}{8}(9z^4 + 70z^2 + 9)\left(\frac{1}{z^2} + \theta\right) \tag{108}$$

$$M_4 = \frac{3}{8}\left[8z^2 + 1 + (z^2 + 8)\,\theta\right] \tag{109}$$

$$M_5 = \frac{1}{8}\left[9z^4 + 70z^2 + 9\right]\theta \tag{110}$$

For constructions for which the shear deformation in the core is negligible, as it is when $\underline{\mu'_{\zeta x}}$ is very large and $\underline{\theta}$ is finite, $\underline{S_x}$ may be taken equal to zero. Then expression (93) can be minimized with respect to $\underline{\eta}$, resulting in:

$$K_o = 2\sqrt{\frac{2M_1M_2\,(I + I_f)}{3\lambda\,(f_1 + f_2)\,h^2}} \tag{111}$$

Thus, $\underline{K_o}$ is a function of $\underline{M_1}$ and $\underline{M_2}$ and the stiffness of the sandwich. It was found by computation that a relative minimum of $M_1M_2 = 0.24$ occurs at $z = 0.95$. The minimum buckling stress is then proportional to:

$$K_o = \frac{4}{5Q_1} \tag{112}$$

where

$$Q_1 = \sqrt{\frac{\lambda\,(f_1 + f_2)\,h^2}{I + I_f}} \tag{113}$$

By letting $N = \dfrac{K}{K_o}$, the following expression can be written from equation (89) for constructions having any value of $\underline{S_x}$:

$$N = \frac{5M_1Q_1}{4\eta} + \frac{5}{6Q_1\,(1 + Q_2)}\left[\frac{M_2\eta + M_3\eta^2S_x}{1 + M_4\eta S_x + M_5\eta^2S_x^{\,2}} + Q_2M_2\eta\right] \tag{114}$$

where

$$Q_2 = \frac{I_f}{I} \tag{115}$$

It was found in Forest Products Laboratory Report No. 1505 ($\underline{10}$) that the values of $\underline{I + I_f}$ and $\underline{I_f}$ can be expressed as follows:

$$I + I_f = \frac{1}{12}\left[h^3 - c^3 - \frac{12cd^2}{1 - c/h}\right] \tag{116}$$

$$I_f = \frac{(h - c)^3}{48} + \frac{(h - c)d^2}{4} \tag{117}$$

where

$$d = \frac{f_1 - f_2}{2} \tag{118}$$

After substituting these expressions in the formulas for $\underline{Q_1}$ and $\underline{Q_2}$ and simplifying $\underline{Q_1}$ and $\underline{Q_2}$ become:

$$Q_1 = \sqrt{\frac{12\lambda\,(1 - c/h)^2}{(1 - c^3/h^3)\,(1 - c/h) - \frac{12cd^2}{h^3}}} \tag{119}$$

$$Q_2 = \frac{\frac{1}{4}(1 - c/h)^4 + 3\,(1 - c/h)^2\,\frac{d^2}{h^2}}{(1 - c^3/h^3)\,(1 - c/h) - \frac{1}{4}(1 - c/h)^4 - 3(1 - c/h)^2\,\frac{d^2}{h^2} - \frac{12cd^2}{h^3}} \tag{120}$$

In equation (114) $\underline{M_1}$, $\underline{M_2}$, $\underline{M_3}$, $\underline{M_4}$, and $\underline{M_5}$ depend upon $\underline{z}$ and $\underline{\theta}$, and $\underline{Q_1}$ and $\underline{Q_2}$, depend upon $\underline{c/h}$ and $\underline{d/h}$. Formula (114) can then be written with appropriate values of $\underline{\theta}$, $\underline{c/h}$, and $\underline{d/h}$ and then a relative minimum value $\underline{N}$ found by choosing a series of values of $\underline{z}$ and $\underline{\eta}$. The facing stress at which buckling will occur is then given by:

$$p_f = \frac{4N}{5Q_1}\,E\,\frac{h}{r} \tag{121}$$

Buckling Stress of Sandwich Constructions with Isotropic Facings of Equal Thickness and Orthotropic or Isotropic Core

Factors in formula (114) can be simplified for sandwich constructions having facings of equal thickness. Then d = 0 and after simplification

$$Q_1 = 2\sqrt{\frac{3\lambda}{c^2/h^2 + c/h + 1}} \tag{122}$$

$$Q_2 = \frac{1}{3}\left(\frac{1 - c/h}{1 + c/h}\right)^2 \tag{123}$$

and finally formula (114) becomes

$$N = \frac{5Q_1}{4}\left\{\frac{M_1}{\eta} + \frac{(1 + c/h)^2}{24\lambda}\left[\frac{M_2\eta + M_3\eta^2 S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x{}^2}\right] + \frac{(1 - c/h)^2}{72\lambda} M_2\eta\right\} \tag{124}$$

The buckling load, which is proportioned to N is obtained by finding the lowest relative minimum of expression (124) with respect to η and z. Expression (124) can be minimized by taking a derivative with respect to η and setting the derivative equal to zero. This leads to a sixth power equation in η. Minimum roots of η with respect to z and for various values of S_x, c/h, and θ were determined by means of a digital computer. Minimum values of N at various values of S_x and for c/h equal to 0.9, 0.8, 0.7, and θ equal to 0.4, 1.0, and 2.5 are given in Table 1 and shown as functions of S_x in figures 4, 5, and 6. Also included in the table and figures are values of N for c/h = 1. These values represent sandwich constructions for which the stiffness of the individual facings are assumed to be zero. Although no actual constructions can be made of this type, the values can be considered as representing the limit for constructions having extremely thin facings. These values of N were obtained as follows. Substitution of c/h = 1 in equation (124) for N leads to

$$N = \frac{5}{2}\sqrt{\lambda}\left[\frac{M_1}{\eta} + \frac{1}{6\lambda}\left(\frac{M_2\eta + M_3\eta^2 S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x{}^2}\right)\right] \tag{125}$$

which has one relative minimum value for $\eta = \infty$. This minimum value is given by

$$N = \frac{5M_3}{12\sqrt{\lambda}\, S_x\, M_5} \tag{126}$$

Substituting in this equation the values of M_3 and M_5 given by equations (108) and (110) yields

$$N = \frac{5\left(\frac{1}{z^2} + \theta\right)}{12\sqrt{\lambda}\; S_x\, \theta} \tag{127}$$

which is minimum for $z = \infty$. This minimum value (for $\sigma = 1/4$; hence $\lambda = 15/16$) is given by

$$N = \frac{5}{3\sqrt{15}\, S_x} = \frac{0.431}{S_x} \tag{128}$$

Substitution of this value of N in equation (118) and using the value of S_x for $f_1 = f_2$ and $c/h = 1$ leads to the following limiting expression for p_f:

$$p_f = \frac{4Eh}{10\sqrt{\lambda}\, r} \cdot \frac{5}{12\sqrt{\lambda}} \cdot \frac{3\lambda\, r\mu' \zeta_x}{Ef} = \frac{h}{2f}\, \mu' \zeta_x \tag{129}$$

For values of S_x ranging from 0 to about 0.6 it was found that equation (128) did not give lowest minimum values. In this range of S_x the minimum values were obtained from equation (124) by use of a digital computer.

The value of N given by equation (128) and the value of the stress given by equation (129) are independent of the radius of the cylinder and are the usual critical values associated with shear instability of the core (15).

The value of θ of 0.4 and its reciprocal 2.5 were used in the calculations because they apply to honeycomb cores oriented with the weak direction and the strong direction parallel to the length of the cylinder. It has been noted from figures 4 and 6 that in the range of small values of S_x where N is independent of c/h, the orientation of the core makes little difference in the value of N.

Application of Theoretical Results

The compressive facing stress at which buckling of cylinders of orthrotropic sandwich construction occurs is given by equation (77).

$$p_f = KE_x \frac{h}{r}$$

where E_x is modulus of elasticity of facings in axial direction, h is sandwich thickness, r is mean radius of curvature, and K is given by formula (79) as

$$K = \frac{64}{3z^2}\left[\frac{\gamma_3}{\eta} - \frac{9\gamma_2^2}{32\gamma_1\eta} + \gamma_4\eta\right]$$

where values of γ_1, γ_2, and γ_3 are functions of z according to equations (80), (81), and (82) and γ_4 is a function of η and z according to equation (86) and K is taken as the least relative minimum with respect to η and z.

For sandwich constructions having isotropic facings of unequal thickness and orthotropic core K is given by

$$K = \frac{4N}{5Q_1}$$

where N is given by equation (114) as

$$N = \frac{5M_1Q_1}{4\eta} + \frac{5}{6Q_1(1 + Q_2)}\left[\frac{M_2\eta + M_3\eta^2S_x}{1 + M_4\eta S_x + M_5\eta^2 S_x{}^2} + Q_2M_2\eta\right]$$

where Q_1 and Q_2 are given by equations (119) and (120) and M_1, M_2, M_3, M_4, and M_5 are functions of z according to equations (106), (107), (108), (109), and (110) and N is taken as the least relative minimum with respect to η and z.

For sandwich constructions having isotropic facings (Poisson's ratio 1/4) of equal thickness and orthotropic core such that $\theta = 0.4$[4] or $\theta = 2.5$[4] or isotropic core ($\theta = 1.0$) equation (114) for N has been solved for $c/h = 1.0$, 0.9, 0.8, and 0.7. Values of N for various S_x values are given in Table 1 and in graphs in figures 4, 5, and 6. Then the critical facing stress is given by

$$p_f = \frac{4N}{5Q_1} E \frac{h}{r}$$

where

$$Q_1 = \frac{3}{2}\sqrt{\frac{5}{c^2/h^2 + c/h + 1}}$$

and N is given in terms of S_x where

[4] These ratios for θ were chosen as representative of honeycomb cores such as were evaluated in Forest Products Laboratory Report No. 1849.

$$S_x = \frac{16cfE}{45rh\mu'_{\zeta x}}$$

and c is core thickness, f is facing thickness, E is modulus of elasticity of facings, h is sandwich thickness, $(c + 2f)$, r is mean radius of curvature, and $\mu'_{\zeta x}$ is modulus of rigidity of core associated with shear strains in the axial-radial plane.

The graphs can be used with little error for determining N for constructions having facings of unequal thickness, provided S_x is calculated using formula (60) and Q_1 is calculated using equation (119).

The analysis may be extended to apply at stresses greater than the proportional limit stress of the facings by use of an appropriate tangent or reduced modulus of elasticity for the facings. This entails a "trial-and-error" solution involving use of the tangent or reduced modulus in the quantity S and elsewhere until the resultant facing stress is compatible with the stress-modulus curve.

Results of the theoretical analysis fall approximately into three zones, depending upon whether there is no shear deformation in the core ($S_x = 0$), some shear deformation in the core (small values of S_x), or considerable shear deformation in the core (large values of S_x).

For no shear deformation, the buckling stress is determined essentially by means of the isotropic or orthotropic theory (depending upon facing properties) with the stiffness determined by considering the spaced facings of the sandwich.

For large shear deformations, the critical stress is associated with instability of the core in shear. This has been observed for sandwich constructions in general (15), and it has been found that the mean critical stress thus determined is the same, regardless of the original assumption of the buckled shape. The smallest value of S_x at which the critical stress is determined by shear instability of the core, however, is greatly affected by the assumed form of the buckled shape. The inclusion of the stiffnesses of the facings I_f gives rise to the family of curves for different values of c/h, as shown in figures 4, 5, and 6, instead of a single curve. If the stiffnesses of the individual facings had been neglected, one curve only, that for $c/h = 1$, would have resulted. The percentage increase in buckling stress due to the stiffnesses of the individual facings increases as the shear deformation increases. For small shear deformations, the increase is negligible.

From the reasoning involved in the theory leading to equation (77), considerable scatter in the experimental values of the critical stress is to be expected because of the effect of initial irregularities. Similar scatter is exhibited by homogeneous cylindrical shells, for the same reason.

The possibility of failure by wrinkling of the facings at a stress lower than that predicted by equation (77) should be considered.

The analysis in this report involves a number of approximations and assumptions. Such procedures are necessary until a more rigorous treatment of the problem is developed. A completely rigorous treatment of the buckling of a homogeneous cylindrical shell is still lacking, in spite of the noteworthy contributions of von Karman and Tsien.

Tests of Curved Panels

The large size of complete circular, cylindrical shells having realistic facing and core thicknesses and curvatures could not be adapted to the available testing apparatus. Therefore, axial compressive tests were conducted on rectangular panels curved to various radii. The dimensions of these panels were chosen so that their widths and lengths were large enough to include at least one buckle of a size predicted by theory ($b > \frac{2\pi r}{n}$ and $a > \frac{2\pi r}{zn}$) as shown in table 2.

It was then assumed that the curved panel would behave approximately as a complete cylinder. The type of edge support (described later) was such as to produce no clamping.

Test Specimens

The test specimens were essentially of isotropic construction having facings of clad 24ST aluminum alloy on cores of either balsa wood, oriented so that the grain direction was normal to the facings, or of corkboard of three different densities. Corkboard cores were chosen, because their low moduli of rigidity afforded means of exploring shells in which sizeable reductions of buckling stresses, caused by large core shear deformations, could easily be obtained. These corkboard cores had shearing moduli of 1,500, 950, and 320 pounds per square inch, as compared to 15,000 pounds per square inch for the end-grain, balsa-wood core.

Dimensions of the specimens are given in table 2. The panel sizes ranged from approximately 70 inches square to panels 12 inches wide and 30 inches long. Mean radii of curvature ranged from approximately 90 inches to 10 inches. The sandwich constructions had facings of 0.012 inch, 0.020 inch, or 0.032 inch thickness on cores of approximately 1/8 inch, 1/4 inch, or 1/2 inch thickness. All constructions tested had facings of equal thickness.

The specimens were manufactured by the bag-molding process. Detailed description of techniques and bonding adhesives used in this process are given in Forest Products Laboratory Report No. 1574 (3). The curvature was attained at the time of molding by using a steel mold curved to the desired radius. A strip of aluminum 1 inch wide and 0.032 inch thick was bonded to the facings at each end of the specimen. This was done to facilitate machining of the specimen ends and also to prevent local end failure during the test. The ends of the specimens were machined square and true in a milling machine.

Testing

The vertical edges of the specimens were held straight by loose-fitting wood guides. These guides were approximately 2 inches by 2 inches in cross section and of lengths 1/4 inch shorter than the test specimen. They were grooved in the lengthwise direction with grooves approximately 1/4 inch deep and wide enough to allow the guides to be slipped onto the edges of the test specimen. No attempt was made to clamp the vertical edges by fitting the guides tightly.

The lower ends of specimens not wider than 30 inches were placed on a heavy flat plate, which was supported by a spherical bearing placed on the lower head of a hydraulic testing machine. The heads of the testing machine were then brought together until the specimen just touched the upper platen with no load indicated. Adjustments were made on the spherical base until no light could be seen between the ends of the specimen and the loading heads. Screw jacks were then placed under the lower loading plate to prevent tilting of the plate while the load was being applied to the specimen. A single thickness of blotting paper was inserted at the ends of the specimen to help prevent local end failures. The load was then applied slowly until failure occurred.

Specimens wider than 30 inches were tested between the heads of a four-screw, mechanically operated, testing machine. No spherical bearing was used. The specimens were cut as true as possible. If light could be seen between the ends of the specimen and the heads of the testing machine, shims of paper or brass were inserted until the gap was closed. These wide specimens were also very long; therefore, small irregularities in the end bearing were absorbed early in the test without causing large variations from uniformity in the stresses in the facings.

Results of Tests

The facing stresses at the failing loads of the curved panels are given in table 2. For later comparison with theoretical values, the parameter N was calculated for each test specimen by using the formula

$$N = \frac{5Q_1 p_f r}{4Eh}$$

where

E = 10,000,000 pounds per square inch (modulus of elasticity of facings)

The visible failures of the specimens were of a type caused by buckling. Large, thin specimens actually showed large buckles, which disappeared after release of load. The appearance of these buckles always caused a sudden drop in the load. Small, thick specimens showed buckling, followed immediately by a crimping appearance at the edges of the buckle. This crimping was undoubtedly due to shear failure of the core caused by high stresses induced in the sandwich by the buckle. Many of the thick specimens exhibited no visible signs of buckling but showed similar crimping. The rapidity of failure occurring immediately upon buckling undoubtedly prevented visual observation of the buckle itself. Similar behavior was observed for cylindrical shells of plywood (11).

Comparison of Theoretical and Experimental Results

The theoretical and experimental values of N are given in table 2. A comparison between them may be obtained by referring to figure 7 which shows the experimental values plotted against the theoretical values. The scatter of points about a line representing equality between experimental and theoretical values shows that theory and experiment agree within approximately ± 30 percent.

In view of the inevitable scatter of experimentally determined buckling stresses that is associated with initial irregularities of shape and variations of material properties, it is concluded that the agreement between results of tests and of theory is satisfactory. The scatter is not as great for shells of sandwich construction as that observed for thin, homogeneous shells (fig. 44, report No. 1322-A (7)) or for plywood shells (fig. 3, Forest Products Laboratory Report No. 1322 (18)). This reduction in scatter may be attributed to a greater total thickness of shell. Thus, irregularities that depart from the true cylindrical surface of the order of the thickness of the shell are less likely to occur in sandwich shells than in thin, homogeneous shells.

Conclusions

The buckling stress of long, thin-walled, circular cylinders of sandwich construction in axial compression can be found with satisfactory accuracy by the formulas and curves of the approximate theoretical analysis of this report.

Curved panels of sizes large enough to include at least one ideal buckle ($b > \frac{2\pi r}{n}$ and $a > \frac{2\pi r}{zn}$) buckle at stresses approximately equal to those of a long, complete cylinder.

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Notation

a	length of buckle.
A	$1/E_b$.
A_1, A_2, A_3, A_4, A_5	defined by equations (42) to (46).
$A'_1, A'_2, A'_3, A'_4, A'_5$	defined by equations (52) to (55).
b	width of buckle.
B	$1/E_a$.
c	thickness of the core.
c_1	mean circumferential stress.
C	$\dfrac{1}{\mu_m} - \dfrac{2\sigma_{xy}}{E_a}$
d	$\dfrac{f_1 - f_2}{2}$
d_1, d_2, d_3	defined by equation (57).
$e_{xx}, e_{xy},$ etc.	components of strain.
e_1, e_2, e_3	defined by equations (26), (27), and (28).
e_4	defined by equation (64).
e_5	defined by equation (66).
E_x, E_y	Young's moduli of the facings
E_a	$\dfrac{E_x (f_1 + f_2)}{h}$
E_b	$\dfrac{E_y (f_1 + f_2)}{h}$
f_1, f_2	thicknesses of the facings.
g	quantity proportional to mean radial expansion.

h	$c + f_1 + f_2$.
I	defined by equation (49).
I_f	defined by equation (50).
k, k'	parameters introduced in equations (31).
K	see equations (74), (75), and (79).
K_1, K_2, K_3	defined by equations (23), (24), and (25).
K_4	defined by equation (84).
n	$2\pi r/b$.
p	mean compressive stress.
p_f	compressive stress in the facings.
q, q'	introduced in equations (31).
r	radius of middle surface of the cylindrical shell.
S_x, S_y	defined by equations (92).
T	defined by equation (83).
u	axial component of displacement.
v	circumferential component of displacement.
U_c	strain energy of the core in the bending of the sandwich shell.
U_F, U_M	strain energy of the facings in the bending of the sandwich shell.
w	radial component of displacement.
W_1	extensional strain energy.
W_2	flexural strain energy.

W_3	virtual work of the compressive load.
W	$(W_1 + W_2 - W_3)/abh$.
X_x, X_y, etc.	components of stress.
z	b/a.
α	π/a.
β	π/b.
γ_1, γ_2, γ_3, γ_4	defined by equations (80), (81), (82), and (86).
δ	a parameter that is proportional to depth of a buckle.
δ_o	initial value of δ.
ϵ	mean compressive strain.
ζ	coordinate shown in fig. 3.
η	$n^2 h/r$.
λ	$(1 - \sigma_{xy} \sigma_{yx})$.
μ_{xy}	modulus of rigidity of the facings.
$\mu'_{\zeta x}$, $\mu'_{y\zeta}$	moduli of rigidity of the core.
μ_m	$\dfrac{\mu_{xy} (f_1 + f_2)}{h}$
ξ	$\delta r/h$.
ξ_o	$\delta_o r/h$.
σ_{xy}, σ_{yx}	Poisson's ratios of the facings.
ϕ	defined by equation (51).
θ	S_y/S_x.

Table 1.--Buckling of sandwich cylinders in axial compression. $p_f = \frac{4N}{5Q_1} E \frac{h}{r}$ N values

S_x	θ = 0.4				θ = 1.0				θ = 2.5			
	c/h = 1.0	c/h = 0.9	c/h = 0.8	c/h = 0.7	c/h = 1.0	c/h = 0.9	c/h = 0.8	c/h = 0.7	c/h = 1.0	c/h = 0.9	c/h = 0.8	c/h = 0.7
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1	.982	.983	.981	.980	.972	.972	.972	.972	.951	.949	.949	.946
.2	.960	.962	.960	.958	.943	.941	.939	.935	.902	.900	.896	.889
.4	.922	.921	.917	.912	.886	.880	.874	.870	.808	.801	.794	.785
.6	.720	.766	.826	.870	.720	.765	.806	.801	.720	.712	.699	.691
.8	.540	.600	.670	.752	.540	.599	.680	.708	.540	.605	.614	.602
1.0	.431	.500	.578	.668	.431	.500	.578	.609	.431	.497	.516	.526
1.2	.358	.431	.514	.611	.358	.431	.506	.540	.358	.426	.449	.466
1.4	.308	.383	.468	.563	.308	.385	.451	.486	.308	.372	.398	.417
1.6	.269	.347	.436	.523	.269	.349	.411	.446	.269	.334	.360	.384
2.0	.216	.296	.387	.460	.216	.298	.350	.386	.216	.276	.306	.332
3.0	.143	.230	.317	.366	.143	.224	.266	.304	.143	.199	.233	.262
4.0	.109	.197	.269	.315	.109	.181	.224	.258	.109	.161	.191	.222
6.0	.073	.161	.215	.258	.073	.138	.175	.210	.073	.120	.151	.185
8.0	.053	.138	.186	.227	.053	.115	.151	.187	.053	.097	.129	.165
10.0	.044	.120	.164	.214	.044	.100	.135	.173	.044	.084	.118	.153

Table 2.--Experimental and theoretical data for curved panels of sandwich construction in axial compression. Cores isotropic

Specimen No.	Facing thickness[1] f	Core thickness c	Sandwich thickness h	Panel width	Panel length	Mean radius of curvature r	Experimental: Facing stress at buckling	Experimental: N	Core shear modulus $\mu'_{\zeta x}$	S_x	c/h	Computed values: N	Computed values: z[2]	Computed values: η[2]	Computed values: Buckle width b[3]	Computed values: Buckle length a[3]
	In.	In.	In.	In.	In.	In.	P.s.i.		P.s.i.							
1155-23	0.012	0.256	0.280	12.1	29.9	22.7	13,100	0.268	950	1.81	0.915	0.306	......	......	......	......
1155-21	.020	.510	.550	20.1	29.9	18.5	14,980	.127	950	3.76	.928	.176	50	0.023	......	......
9114-23	.012	.263	.287	12.1	29.9	23.4	6,660	.137	320	5.22	.916	.156	81	.007	......	......
1155-18	.020	.511	.551	12.1	30.0	11.2	16,190	.082	950	6.20	.928	.122	1.9	6.94	5.92	3.12
9114-17	.020	.516	.556	12.1	29.9	23.8	9,330	.100	320	8.66	.928	.097	1.7	7.19	8.52	5.01
9114-21	.020	.515	.555	20.1	29.7	18.8	9,590	.081	320	10.98	.929	.082	1.6	7.34	7.47	4.67
9114-18	.020	.508	.548	12.0	29.9	11.6	11,470	.061	320	17.76	.927	.062	1.4	7.21	5.89	4.21
ANC 113-1	.020	.484	.524	12.1	30.0	11.0	11,200	.059	320	18.64	.924	.061	1.4	6.96	5.72	4.09
ANC 113-2	.020	.500	.540	12.1	30.0	10.9	11,810	.060	320	18.83	.926	.061	1.4	7.11	5.72	4.09
61-1	.012	.133	.157	69.2	70.0	92.9	7,100	1.102	15,000	.026	.848	.994	.95	.159	60.2	63.4
61-3	.012	.129	.153	69.2	69.9	73.2	8,300	1.046	15,000	.033	.844	.990	.96	.160	52.5	54.7
1463-10	.020	.262	.302	11.8	29.2	22.5	12,510	.242	1,500	1.82	.868	.343	2.5	3.08	9.29	3.72
1463-14	.020	.259	.299	20.1	29.4	19.3	13,070	.219	1,500	2.12	.866	.307	2.2	3.36	8.23	3.74
1155-10	.020	.254	.294	12.1	30.0	24.4	11,660	.252	950	2.65	.864	.263	2.0	3.53	8.91	4.45
9114-13	.020	.262	.302	20.0	29.9	66.4	4,740	.270	320	2.90	.868	.245	1.9	3.72	14.56	7.66
1155-14	.020	.255	.295	20.1	29.9	19.4	10,320	.176	950	3.34	.864	.224	1.8	3.69	7.79	4.33
1463-11	.020	.262	.302	11.8	29.4	10.6	16,810	.153	1,500	3.88	.868	.200	1.7	3.87	5.72	3.36
1155-11	.020	.252	.292	12.1	29.9	10.5	12,380	.115	950	6.15	.863	.150	1.5	3.81	5.61	3.74
9114-10	.020	.249	.289	12.0	29.7	26.2	6,680	.157	320	7.30	.861	.135	1.4	3.76	8.92	6.37
9114-14	.020	.257	.297	20.1	29.6	18.7	6,770	.110	320	10.30	.865	.110	1.3	3.86	7.47	5.75
9114-11	.020	.257	.297	12.1	30.0	11.3	8,420	.083	320	17.00	.865	.085	1.2	3.83	5.89	4.91
1463-25	.032	.264	.328	11.8	29.4	22.9	12,170	.229	1,500	2.67	.805	.286	1.7	2.57	10.74	6.31
1155-25	.032	.256	.320	12.0	29.9	23.2	12,120	.237	950	4.12	.800	.218	1.5	2.58	10.68	7.12
9114-25	.032	.261	.325	12.1	29.8	27.2	7,480	.168	320	10.48	.804	.130	1.2	2.63	11.49	9.58
1155-6	.020	.145	.185	20.1	29.9	60.5	6,370	.563	950	.97	.784	.594	2.2	1.88	15.26	6.94
1463-3	.020	.134	.174	11.8	29.4	23.6	9,600	.354	1,500	1.54	.770	.433	1.8	2.04	8.92	4.96
1463-7	.020	.135	.175	20.1	29.3	19.0	9,020	.268	1,500	1.92	.772	.370	1.7	2.13	7.85	4.62
1155-3	.020	.137	.177	12.1	29.9	27.7	7,220	.310	950	2.09	.774	.349	1.7	2.17	9.42	5.54
9114-6	.020	.128	.168	20.0	29.8	58.6	3,310	.319	320	2.89	.762	.287	1.5	2.12	13.50	9.00
1155-7	.020	.135	.175	20.1	29.9	18.6	8,340	.243	950	3.11	.772	.270	1.5	2.23	7.54	5.03
1463-4	.020	.135	.175	11.8	29.4	10.5	13,670	.225	1,500	3.48	.772	.252	1.5	2.24	5.71	3.80
1155-4	.020	.135	.175	12.1	30.0	10.8	10,450	.177	950	5.35	.772	.196	1.3	2.25	5.75	4.42
9114-3	.020	.135	.175	12.1	30.0	26.0	5,620	.230	320	6.59	.772	.176	1.3	2.27	8.85	6.81
9114-7	.020	.125	.165	20.1	29.8	18.4	5,820	.179	320	9.15	.759	.154	1.2	2.13	7.47	6.23
9114-4	.020	.143	.183	12.1	29.9	11.3	8,580	.145	320	15.35	.781	.118	1.1	2.33	5.92	5.38

[1] All facings were of equal thickness and of clad 2024-T3 aluminum alloy.

[2] These values of z and η produced minimum theoretical values of N.

[3] $b = 2\pi\sqrt{\frac{rh}{\eta}} \qquad a = \frac{b}{z}$

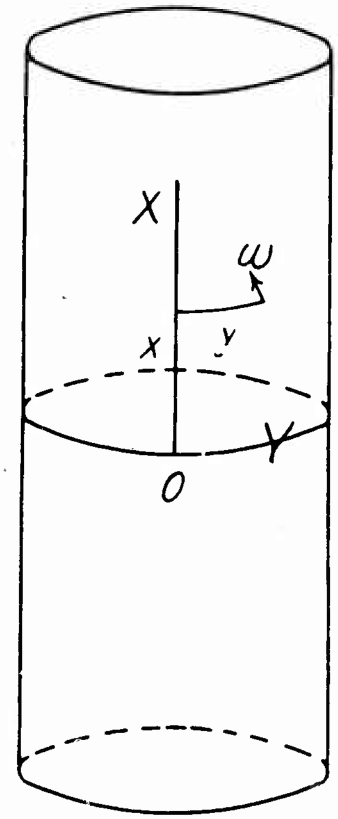

Figure 1.--Choice of coordinates on the surface of a cylinder.

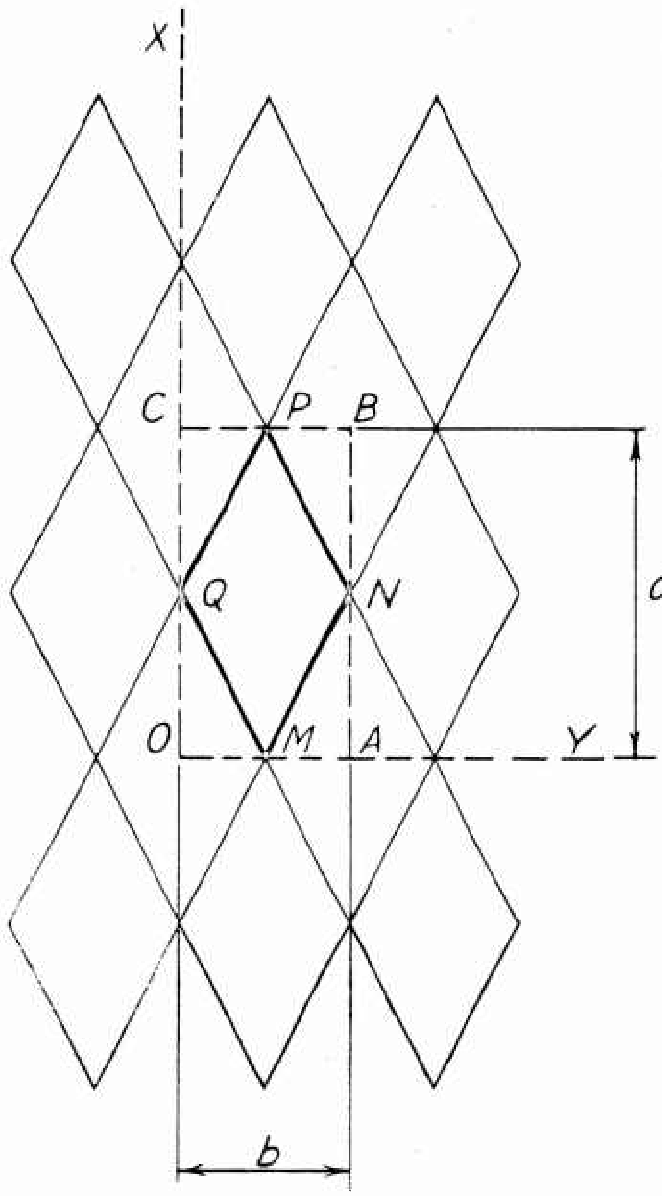

Figure 2. --Nodal lines of assumed diamond-shaped buckling pattern.

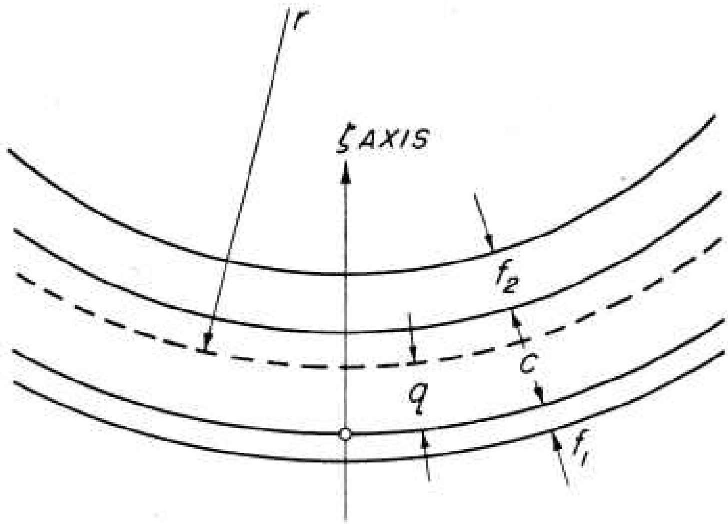

Figure 3. --Section of a cylindrical shell, where r is the radius of the middle surface of the shell, c is the thickness of the core, f_1 and f_2 are the thicknesses of the facings, q is the distance indicated in the figure, and ζ is the coordinate indicated in the figure.
Note: The curved lines are arcs of concentric circles.

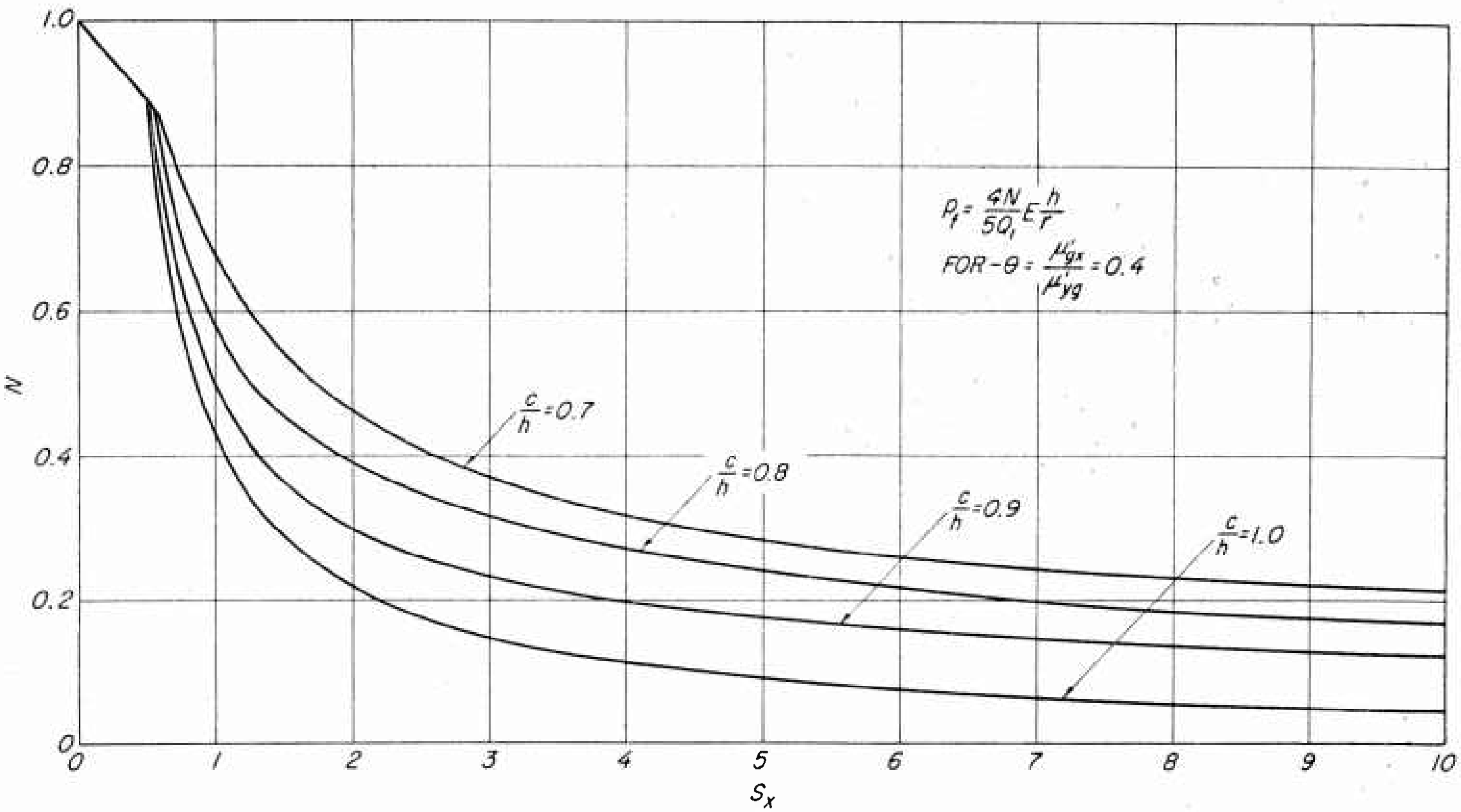

Figure 4. --N values for buckling of cylinders of sandwich constructions in axial compression. Isotropic sandwich facings of equal thickness, orthotropic core.

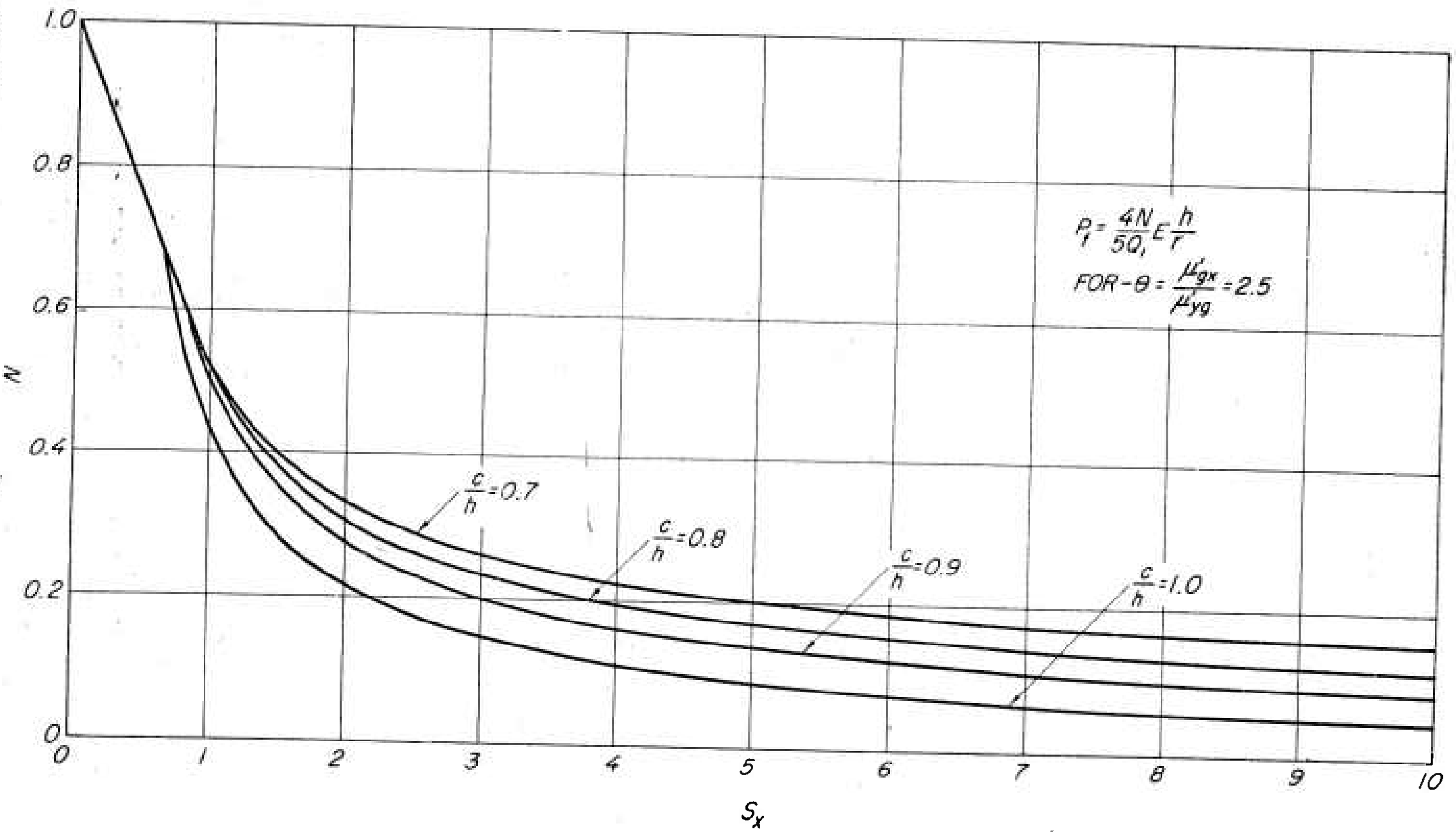

Figure 5. --N values for buckling of cylinders of sandwich constructions in axial compression. Isotropic sandwich facings of equal thickness, orthotropic core.

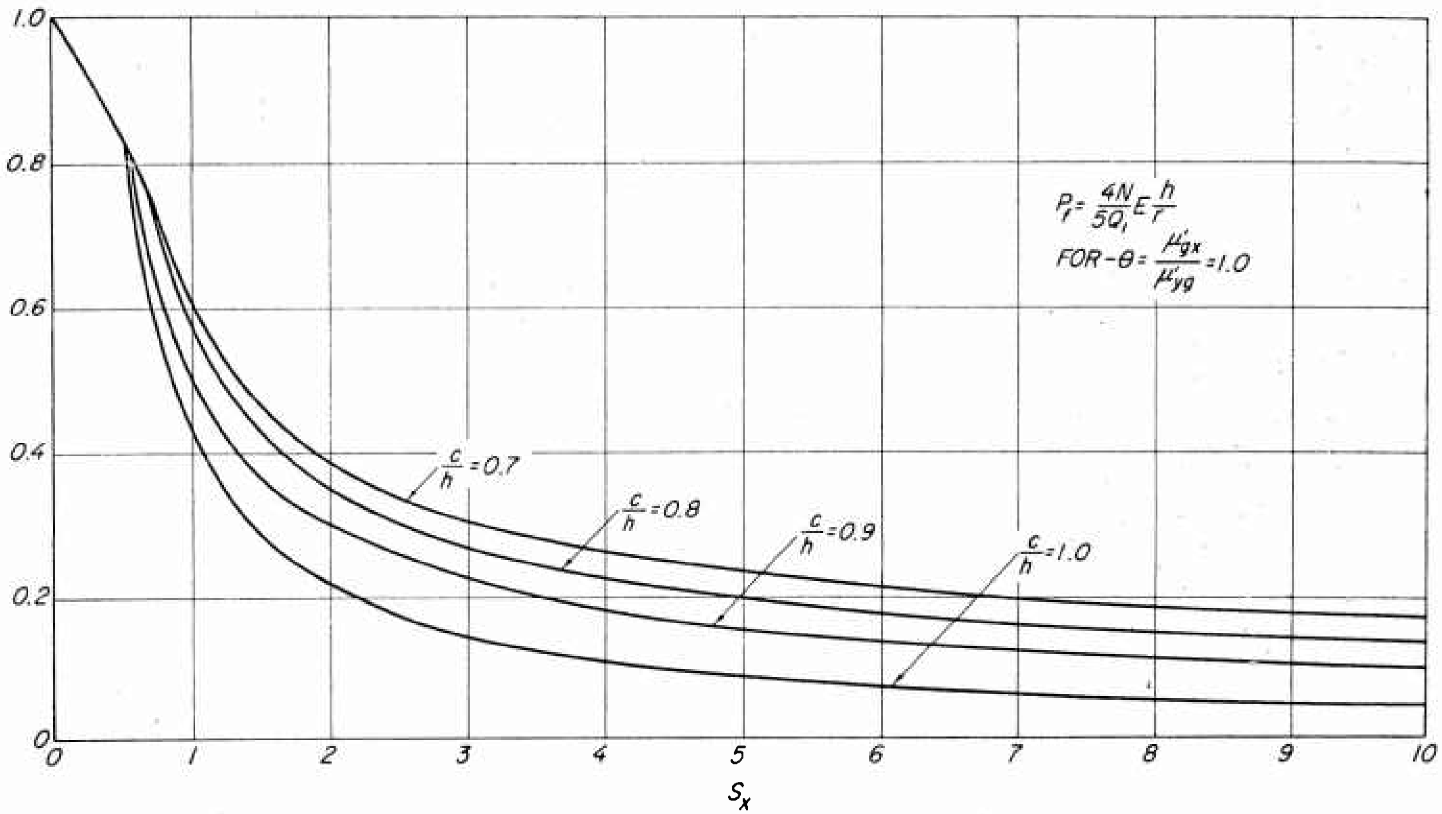

Figure 6. --N values for buckling of cylinders of isotropic sandwich constructions in axial compression. Sandwich facings of equal thickness.

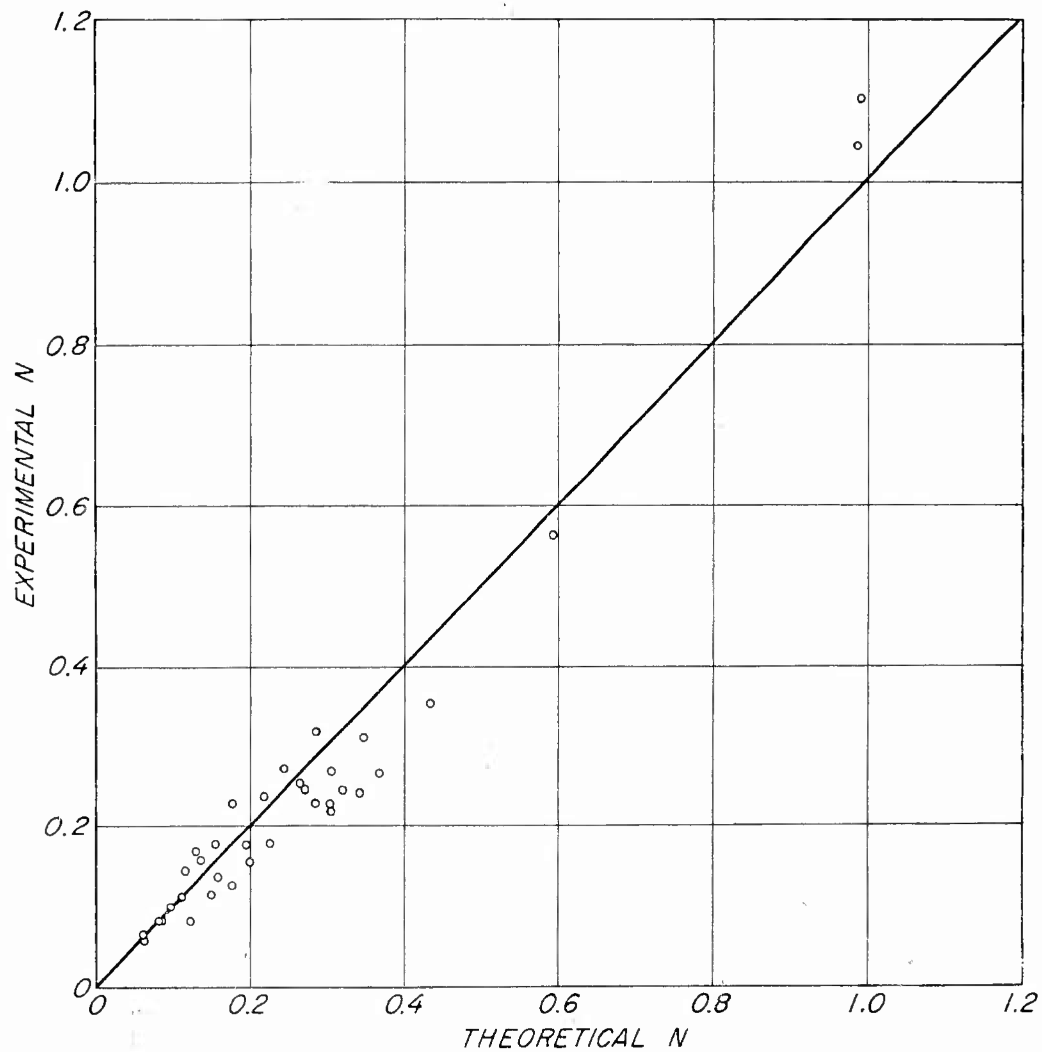

Figure 7. --Comparison of experimental values of N with theoretical values.

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SECTION III

FLEXURE AND TORSION OF COMPOSITE CYLINDERS

By

W. S. ERICKSEN

Air Force Institute of Technology
For Materials Laboratory

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Summary

A solution to the problem of determining the components of stress and displacement in composite cylinders supported as cantilever beams and subjected to flexural and torsional loads is given in this report. The type of cylinder considered is that composed of three circular, co-axial, layers of different materials that are bonded at their junctures. Two main cases are considered; one in which the material in the center layer is cylindrically aeolotropic, the other in which it is isotropic. The material in the inner and outer layers is assumed to be isotropic. By taking the thickness of one of the layers equal to zero, the results are applicable to a two-layer cylinder and, by making the inner radius zero, they are applicable to a solid cylinder.

LIST OF ABBREVIATIONS

The following is a partial list of the symbols used in this report:

E_m: Young's modulus of an isotropic material.

E_z, E_r, E_θ: Young's modulii of a cylindrically aeolotropic material.

$EI = E_1 I_1 + E_2 I_2 + E_3 I_3$ if intermediate layer is isotropic

$= E_1 I_1 + E_z I_2 + E_3 I_3$ if intermediate layer is cylindrically aeolotropic.

$e_{zz}, \ldots, e_{z\theta}$: components of strain.

$$F_1 = \frac{2(1 - \sigma_{\theta z}\sigma_{z\theta})E_z(3 - \mu)}{(\sigma_{z\theta} - \sigma_{zr})E_\theta} + 2(\sigma_{zr} + 3\sigma_{z\theta})$$

$$F_n = (n + 1)\left\{\sigma_{zr} + (n + 2)\sigma_{z\theta}\right\} \qquad n = i, j$$

G_m Shear modulus of an isotropic material.

$G_{r\theta}$, $G_{\theta z}$, G_{zr}: Shear modulii of a cylindrically aeolotropic material.

$GI = G_1 I_1 + G_2 I_2 + G_3 I_3$ if intermediate layer is isotropic

$= G_1 I_1 + G_{\theta z} I_2 + G_3 I_3$ if intermediate layer is cylindrically aeolotropic.

$$H_1 = \frac{2(1 - \sigma_{\theta z}\sigma_{z\theta})(3 - \mu)(\sigma_{zr} - 2\sigma_{z\theta})}{E_\theta(\sigma_{z\theta} - \sigma_{zr})} + 12\frac{(1 - \sigma_{\theta z}\sigma_{z\theta})}{E_\theta}$$

List of Abbreviations (Cont'd)

$$-2\left\{\frac{1-\sigma_{rz}\,\sigma_{zr}}{E_r}-\frac{\sigma_{\theta r}+\sigma_{\theta z}\,\sigma_{zr}}{E_\theta}+\frac{1}{G_{r\theta}}\right\}$$

$$H_n = (n+1)^2(n+2)\left\{\frac{1-\sigma_{\theta z}\,\sigma_{z\theta}}{E_\theta}\right\}-(n+1)\left\{\frac{1-\sigma_{rz}\sigma_{zr}}{E_r}\right.$$

$$\left.-\frac{\sigma_{\theta r}+\sigma_{\theta z}\,\sigma_{z\theta}}{E_\theta}+\frac{1}{G_{r\theta}}\right\}$$

$n = i, j$

$$I_m = \frac{\pi}{4}\left(r_m^4 - r_{m-1}^4\right)$$

$$i = (1+\mu)^{\frac{1}{2}} - 1$$

$$j = -(1+\mu)^{\frac{1}{2}} - 1$$

$$k = \left(\frac{G_{\theta z}}{G_{zr}}\right)^{\frac{1}{2}}$$

$$\left.\begin{aligned} K_{1n} &= \frac{1}{E_z}F_n - (n+1)\frac{\sigma_{zr}}{E_r} - (n+1)(n+2)\frac{\sigma_{\theta z}}{E_\theta} \\ K_{2n} &= -\frac{\sigma_{zr}F_n}{E_z} + \frac{(n+1)}{E_r} - (n+1)(n+2)\frac{\sigma_{\theta r}}{E_\theta} \\ K_{3n} &= -\frac{\sigma_{z\theta}}{E_z}F_n - \frac{(n+1)\sigma_{r\theta}}{E_r} + \frac{(n+1)(n+2)}{E_\theta} \end{aligned}\right\} \quad n = 1, i, j$$

ℓ : length of cylinder

m : index denoting 1, 2 or 3

M : applied torsional moment

List of Abbreviations (Cont'd)

n : index that ranges over the values 1, i, j.

P : applied radial force.

r_m : outer radius of layer m.

u'_z, u'_r, u'_θ : components of displacement associated with derived stress components.

u^*_z, u^*_r, u^*_θ : components of rigid body displacement.

$u_z = u'_r + u^*_r,\ u_r = u'_r + u^*_r,\ u_\theta = u'_\theta + u^*_\theta$, components of displacement.

z, r, θ : cylindrical coordinates.

$$\mu = \left\{ \frac{1 - \sigma_{rz}\sigma_{zr}}{E_r} - 2\,\frac{(\sigma_{\theta r} + \sigma_{\theta z}\ \sigma_{zr})}{E_\theta} + \frac{1}{G_{r\theta}} \right\} \frac{E_\theta}{1 - \sigma_{\theta z}\sigma_{z\theta}}$$

$$\rho_m = \frac{r_{m-1}}{r_m}$$

σ_m : Poisson's ratio of an isotropic material.

σ : The value of Poisson's ratio when this ratio is assumed the same for each layer.

$\sigma_{zr}, \sigma_{r\theta}, \ldots, \sigma_{\theta z}$: Poisson's ratios of a cylindrically aeolotropic material.

$\sum_n$: A symbol denoting summation over the indices n.

$\tau_{zz}, \ldots, \tau_{\theta z}$: Components of stress.

INTRODUCTION

This report deals with the problem of determining the components of stress and displacement in composite circular cylinders under flexure and torsion. The composite cylinders under consideration are composed of three co-axial layers of different materials. These layers are bonded together at their junctures so that at the junctures the components of stress and displacement are continuous. The materials of which the inner and outer layers of the cylinders are composed are assumed to be isotropic while the material of the intermediate layer is treated in one case as isotropic and in the second as cylindrically aeolotropic.[1]

The results developed here are applicable to cylinders that are long in comparison with their diameters. This restriction is usual for solutions of the present type. There are no further restrictions on the dimensions of the cylinder or on the relative thicknesses of the various layers. Thus the results cover the range from a cable-like cylinder with a solid core to a hollow cylinder with walls that are thin. Also, by taking the thickness of one of the extreme layers as zero, the results apply to a cylinder composed of two layers.

The solutions of the differential equations of equilibrium and compatibility that are used as a basis for the analysis have been previously derived.[1] For the case of three isotropic layers results could have been obtained by methods developed by Muskhelishvili and others.[2] The explicit results given by Muskhelishvili, those for a cylinder of two isotropic layers with equal Poisson's ratios, have been compared with the present results for this case as a means of checking the formulas developed here.

By way of arrangement, this report is divided into four parts, the first of which contains an enumeration of the equations of equilibrium and compatibility, the boundary conditions and the conditions imposed upon the stress resultants and resultant moments over sections of the cylinder perpendicular to the axis. The solution of the problem for the case of an isotropic intermediate layer is given in Part 2 and Part 3 contains the solution for the case of a cylindrically aeolotropic intermediate layer. In Part 4 the results obtained under the assumption of equal Poisson's ratios for the materials of the three layers are given for the cases considered in Parts 2 and 3. These results are considerably simpler than those for the general cases and may serve for approximate use.

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[1] Bending and Torsion of Circular Cylinder Cantilever Beams of Cylindrically Aeolotropic Material by W. S. Ericksen, Journal of Applied Mechanics, June 1956.

[2] Some Basic Problems of the Theory of Elasticity by N. J. Muskhelishvili, P. Noordhoff Ltd Groningen Holland 1953.

The writing of this report is motivated mainly by the applicability of the results to sandwich cylinders. In this application it is possible that simplified formulas may be obtained if the intermediate, or core layer, is thick and of low density in comparison with the outer layers. Simplifications should also result if the thickness of each layer is small compared with some standard radius, say the radius of the mid-surface of the cylinder. These simplifications are left to be made, where possible, in individual applications.

1. FORMULATION OF THE PROBLEM.

Let the axes of reference for a cylindrical coordinate system (z, r, θ) be fixed in the cylinder with the longitudinal axis in coincidence with the axis of symmetry of the cylinder as shown in Figure 1. As indicated in the figure the symbols r_0 and r_3 designate the inner and outer radii of the cylinder, respectively, while r_1 and r_2 denote the radii at the junctures of the layers. The layers are numbered 1, 2, and 3, according to the convention that the one for which

$$r_{m-1} \leq r \leq r_m$$

is layer m. These layer numbers are used as subscripts with various symbols to associate the quantity designated by the symbol with a given layer.

In the preceding notation the equations of equilibrium are given as follows:

$$\frac{\partial \tau_{rrm}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta m}}{\partial \theta} + \frac{\partial \tau_{rzm}}{\partial z} + \frac{\tau_{rrm} - \tau_{\theta\theta m}}{r} = 0 \qquad [1]$$

$$\frac{1}{r}\frac{\partial \tau_{\theta\theta m}}{\partial \theta} + \frac{\partial \tau_{r\theta m}}{\partial r} + \frac{\partial \tau_{\theta zm}}{\partial z} + \frac{2}{r}\tau_{r\theta m} = 0 \qquad [2]$$

$$\frac{\partial \tau_{zrm}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta zm}}{\partial \theta} + \frac{\partial \tau_{zzm}}{\partial z} + \frac{1}{r}\tau_{rzm} = 0, \qquad [3]$$

$$m = 1, 2, 3$$

The compatibility conditions are

$$\frac{\partial^2 e_{rrm}}{\partial z^2} + \frac{\partial^2 e_{zzm}}{\partial r^2} - \frac{\partial^2 e_{rzm}}{\partial r \partial z} = 0 \qquad [4]$$

$$\frac{1}{r^2}\frac{\partial^2 e_{zzm}}{\partial \theta^2} + \frac{\partial^2 e_{\theta\theta m}}{\partial z^2} - \frac{2}{r}\frac{\partial^2 e_{\theta zm}}{\partial \theta \partial z} + \frac{1}{r}\frac{\partial e_{zzm}}{\partial r} - \frac{2}{r}\frac{\partial e_{rzm}}{\partial z} = 0 \qquad [5]$$

$$\frac{\partial}{\partial z}\left[\frac{\partial e_{r\theta m}}{\partial z} - \frac{\partial e_{\theta zm}}{\partial r} - \frac{1}{r}\left(\frac{\partial e_{zrm}}{\partial \theta} - e_{\theta zm}\right)\right] + \frac{1}{r}\frac{\partial^2 e_{zzm}}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial e_{zzm}}{\partial \theta} = 0 \qquad [6]$$

$$\frac{\partial}{\partial r}\left[\frac{\partial (r\, e_{\theta\theta m})}{\partial r} - \frac{\partial e_{r\theta m}}{\partial \theta} - e_{rrm}\right] + \frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial e_{rrm}}{\partial \theta} - \frac{\partial e_{r\theta m}}{\partial r} - \frac{2}{r} e_{r\theta m}\right] = 0 \qquad [7]$$

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial (r\, e_{\theta zm})}{\partial r} - \frac{\partial e_{zrm}}{\partial \theta}\right]\right\} = -\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial e_{rrm}}{\partial \theta} - \frac{\partial e_{r\theta m}}{\partial r} - \frac{2}{r} e_{r\theta m}\right] \qquad [8]$$

$$\frac{\partial}{\partial \theta}\left\{\frac{1}{r}\left[\frac{\partial (r\, e_{\theta zm})}{\partial r} - \frac{\partial e_{zrm}}{\partial \theta}\right]\right\} = \frac{\partial}{\partial z}\left[\frac{\partial (r\, e_{\theta\theta m})}{\partial r} - \frac{\partial e_{r\theta m}}{\partial \theta} - e_{rrm}\right] \qquad [9]$$

$m = 1, 2, 3.$

Let the cylinder be fixed[3] at the base $z = 0$, and be subjected to a force P directed along the radius $\theta = 0$ and a torsional moment M at $z = \ell$. The conditions imposed upon the stress resultants and couples are then

$$\sum_{m=1}^{3} \int_{r_{m-1}}^{r_m} \int_0^{2\pi} (\tau_{zrm} \cos\theta - \tau_{\theta zm} \sin\theta) r \, dr \, d\theta = P \qquad [10]$$

$$\sum_{m=1}^{3} \int_{r_{m-1}}^{r_m} \int_0^{2} \tau_{zzm} \, r^2 \cos\theta \, dr \, d\theta = -P(\ell - z) \qquad [11]$$

$$\sum_{m=1}^{3} \int_{r_{m-1}}^{r_m} \int_0^{2\pi} \tau_{\theta zm} \, r^2 \, dr \, d\theta = M \qquad [12]$$

$$\sum_{m=1}^{3} \int_{r_{m-1}}^{r_m} \int_0^{2\pi} (\tau_{zrm} \sin\theta + \tau_{\theta zm} \cos\theta) r \, dr \, d\theta = 0 \qquad [13]$$

$$\sum_{m=1}^{3} \int_{r_{m-1}}^{r_m} \int_0^{2\pi} \tau_{zzm} \, r^2 \sin\theta \, dr \, d\theta = 0 \qquad [14]$$

$$\sum_{m=1}^{3} \int_{r_{m-1}}^{r_m} \int_0^{2\pi} \tau_{zzm} \, r \, dr \, d\theta = 0 \qquad [15]$$

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[3] With limitations discussed in Section 4.

The boundary conditions to be imposed are that the outer and inner curved surfaces of the cylinder are free of stress and that the components of stress and displacement are continuous at the junctures of the layers. Thus if the symbol $\tau_m(z, r, \theta)$ designates any of the stress components τ_{rrm}, $\tau_{r\theta m}$, or τ_{zrm} and if $u_m(z, r, \theta)$ designates any one of the components of displacement u_{zm} u_{rm} or $u_{\theta m}$ the conditions imposed are

$$\left.\begin{aligned}
&\tau_1\,(z, r_o, \theta) = 0\\
&\tau_1\,(z, r_1, \theta) = \tau_2\,(z, r_1, \theta)\\
&\tau_2\,(z, r_2, \theta) = \tau_3\,(z, r_2, \theta)\\
&\tau_3\,(z, r_3, \theta) = 0
\end{aligned}\right\} \qquad [16]$$

and

$$\left.\begin{aligned}
&u_1\,(z, r_1, \theta) = u_2\,(z, r_1, \theta)\\
&u_2\,(z, r_2, \theta) = u_3\,(z, r_2, \theta)
\end{aligned}\right\} \qquad [17]$$

2. RESULTS FOR THREE ISOTROPIC LAYERS.

The stress-strain relations for the case of three isotropic layers are written

$$\left.\begin{aligned}
e_{zzm} &= \frac{1}{E_m}\left\{\tau_{zzm} - \sigma_m \tau_{rrm} - \sigma_m \tau_{\theta\theta m}\right\} \\
e_{rrm} &= \frac{1}{E_m}\left\{-\sigma_m \tau_{zzm} + \tau_{rrm} - \sigma_m \tau_{\theta\theta m}\right\} \\
e_{\theta\theta m} &= \frac{1}{E_m}\left\{-\sigma_m \tau_{zzm} - \sigma_m \tau_{rrm} + \tau_{\theta\theta m}\right\} \\
e_{r\theta m} &= \frac{(1+\sigma_m)}{E_m}\tau_{r\theta m} \\
e_{\theta zm} &= \frac{1}{2G_m}\tau_{\theta zm} \\
e_{zrm} &= \frac{1}{2G_m}\tau_{zrm}
\end{aligned}\right\} \qquad [18]$$

$$m = 1,\ 2,\ 3.$$

The components of stress given by the following expressions satisfy the equations of equilibrium [1], [2] and [3], and by the use of relations [18] they also satisfy the compatibility equations [4] to [9] inclusive.

$$\tau_{zzm} = (8\sigma_m A_m + C_m)\, r(\ell - z) \cos\theta \ldots \qquad [19]$$

$$\tau_{rrm} = (2A_m r - 2\frac{B_m}{r^3})(\ell - z) \cos\theta \ldots \qquad [20]$$

$$\tau_{\theta\theta m} = (6A_m r + \frac{2B_m}{r^3})(\ell - z) \cos\theta \ldots \qquad [21]$$

$$\tau_{r\theta m} = (2A_m r - \frac{2B_m}{r^3})(\ell - z) \sin\theta \ldots \qquad [22]$$

$$\tau_{zrm} = \left[J_m + \frac{L_m}{r^2} + A_m \left\{ 3\sigma_m + \frac{2G_m}{E_m}(1 - \sigma_m^2) \right\} r^2 \right.$$

$$\left. + \frac{C_m}{8E_m}(3E_m - 2\sigma_m G_m) r^2 \right] \cos\theta \; . \; . \; . \qquad [23]$$

$$\tau_{\theta zm} = \left[-J_m + \frac{L_m}{r^2} - 3A_m \left\{ \frac{\sigma_m}{3} + \frac{2G_m}{E_m}(1 - \sigma_m^2) \right\} r^2 \right.$$

$$\left. \frac{-3C_m}{8E_m}\left(\frac{E_m}{3} - 2\sigma_m G_m \right) r^2 \right] \sin\theta + D_m G_m r \; . \; . \; . \; . \qquad [24]$$

These particular components of stress prove to be satisfactory for the present loading conditions. They may be derived by the methods of Section 3 of Reference 1. However, the fact that they satisfy the conditions of equilibrium and compatibility is readily verified by direct substitution.

When the components of stress are of the preceding forms the components of displacement are given by the expressions that follow. In these expressions each component is separated into two parts, the first of which is primed and designates the component directly associated with the given stress components, while the second part is starred and represents a rigid body displacement. Thus

$$u_{zm} = u'_{zm} + u^*_{zm} \; . \; . \; . \qquad [25]$$

$$u_{rm} = u'_{rm} + u^*_{rm} \; . \; . \; . \qquad [26]$$

$$u_{\theta m} = u'_{\theta m} + u^*_{\theta m} \; . \; . \; . \qquad [27]$$

where

$$u'_{zm} = \left[-\frac{C_m}{2E_m} r (\ell - z)^2 + \frac{J_m}{G_m} r \right.$$

$$+\left\{\frac{A_m}{E_m}\left[(1-2\sigma_m)(1+\sigma_m)+\frac{\sigma_m E_m}{G_m}\right]-\frac{C_m}{8E_m}\left[2\sigma_m-\frac{E_m}{G_m}\right]\right\}r^3$$

$$\left.-\left\{\frac{B_m(1+\sigma_m)}{E_m}+\frac{L_m}{G_m}\right\}\frac{1}{r}\right]\cos\theta \qquad [28]$$

$$u'_{rm}=\left[-\frac{C_m}{6E_m}(\ell-z)^3+\left\{\frac{A_m}{E_m}(1-4\sigma_m)(1+\sigma_m)-\frac{\sigma_m C_m}{2E_m}\right\}r^2(\ell-z)\right.$$

$$\left.+\frac{B_m(1+\sigma_m)}{E_m r^2}(\ell-z)\right]\cos\theta \qquad [29]$$

$$u'_{\theta m}=\left[\frac{C_m}{6E_m}(\ell-z)^3+\left\{\frac{A_m}{E_m}(5-4\sigma_m)(1+\sigma_m)-\frac{\sigma_m C_m}{2E_m}\right\}r^2(\ell-z)\right.$$

$$\left.\frac{+B_m(1+\sigma_m)(\ell-z)}{E_m r^2}\right]\sin\theta+D_m rz \qquad [30]$$

and

$$u^*_{zm}=-\beta_m r\cos\theta-\delta_m r\sin\theta+\mathcal{J}_m \qquad [31]$$

$$u^*_{rm}=(\alpha_m+\beta_m z)\cos\theta+(\gamma_m+\delta_m z)\sin\theta \qquad [32]$$

$$u^*_{\theta m}=-(\alpha_m+\beta_m z)\sin\theta+(\gamma_m+\delta_m z)\cos\theta+\epsilon_m r. \qquad [33]$$

The constants appearing in expressions [19] through [24] and in [28], [29] and [30] are determined explicitly by conditions [11], [12], [16] and [17] as follows:

$$A_1=\frac{P\varphi_1}{\Delta}\left[(\sigma_2-\sigma_1)\varphi_2 I_2+(\sigma_3-\sigma_1)\varphi_3 I_3\right] \qquad [34]$$

$$A_2=\frac{P\varphi_2}{\Delta}\left[(\sigma_1-\sigma_2)\varphi_1 I_1+(\sigma_3-\sigma_2)\varphi_3 I_3\right] \qquad [35]$$

$$A_3 = \frac{P\varphi_3}{\Delta}\left[(\sigma_1 - \sigma_3)\varphi_1 I_1 + (\sigma_2 - \sigma_3)\varphi_2 I_2\right] \quad [36]$$

$$B_1 = A_1 r_0^4 \quad [37]$$

$$B_2 = A_2 r_1^2 - A_1(r_1^4 - r_0^4) \quad [38]$$

$$B_3 = A_3 r_3^4 \quad [39]$$

$$\frac{C_1}{E_1} = \frac{C_2}{E_2} = \frac{C_3}{E_3} = -\frac{P}{\Delta}(\varphi_1 I_1 + \varphi_2 I_2 + \varphi_3 I_3) \quad [40]$$

$$D_1 = D_2 = D_3 = \frac{M}{2(G_1 I_1 + G_2 I_2 + G_3 I_3)} \quad [41]$$

$$I_m = \frac{\pi}{4}(r_m^4 - r_{m-1}^4) \quad [42]$$

$$\Delta = \left[\varphi_1 I_1 + \varphi_2 I_2 + \varphi_3 I_3\right]\left[E_1 I_1 + E_2 I_2 + E_3 I_3\right]$$
$$+ 8\Big[\varphi_1 I_1 \varphi_2 I_2(\sigma_2 - \sigma_1)^2 + \varphi_2 I_2 \varphi_3 I_3(\sigma_2 - \sigma_3)^2 +$$
$$+ \varphi_3 I_3 \varphi_1 I_1(\sigma_3 - \sigma_1)^2\Big] \quad [43]$$

$$\frac{1}{\varphi_m} = \frac{8(1 - \sigma_m^2)}{E_m} - 2\left\{1 - \rho_m^{4(2-m)}\right\}\left\{\frac{1 + \sigma_m}{E_m} - \frac{1 + \sigma_2}{E_2}\right\} \quad [44]$$

and

$$\rho_m = \frac{r_{m-1}}{r_m}. \quad [45]$$

Having made these determinations the following expressions may be evaluated:

$$J_2 = \frac{r_1^4 \mathcal{T}_1 \lambda_3 - r_2^4 \mathcal{T}_3 \lambda_1}{r_1^2 \Omega_1 \lambda_3 - r_2^2 \Omega_3 \lambda_1} \qquad [46]$$

$$L_2 = \frac{r_1^2 r_2^2 \left\{ r_2^2 \mathcal{T}_3 \Omega_1 - r_1^2 \mathcal{T}_1 \Omega_3 \right\}}{r_1^2 \Omega_1 \lambda_3 - r_2^2 \Omega_3 \lambda_1} \qquad [47]$$

$$J_1 = \frac{1}{r_1^2 - r_0^2} \left[J_2 r_1^2 + L_2 + N_2 r_1^4 \right] - N_1 (r_1^2 + r_0^2) \qquad [48]$$

$$L_1 = -J_1 r_0^2 - N_1 r_0^4 \qquad [49]$$

$$J_3 = \frac{1}{r_2^2 - r_3^2} \left[J_2 r_2^2 + L_2 + N_2 r_2^4 \right] - N_3 (r_2^2 + r_3^2) \qquad [50]$$

$$L_3 = - J_3 r_3^2 - N_3 r_3^4 \qquad [51]$$

where

$$N_m = \frac{A_m}{E_m} \left\{ 3 \sigma_m E_m + 2G_m (1 - \sigma_m^2) \right\} + \frac{C_m}{8E_m} \left\{ 3E_m - 2 \sigma_m G_m \right\} \qquad [52]$$

$$\mathcal{T}_m = N_m \left\{ 1 - \rho_m^{2(2-m)} \right\} \left\{ 1 + 2\rho_m^{2(2-m)} \right\} - N_2 \left\{ 1 + \rho_m^{2(2-m)} \right\}$$
$$- G_m \left\{ 1 - \rho_m^{2(2-m)} \right\} \left\{ \psi_m - \psi_2 \right\} \qquad [53]$$

$$\psi_m = \frac{A_m}{E_m}\left\{2(1-\sigma_m^2) + \frac{\sigma_m E_m}{G_m}\right\} + A_m\left\{\frac{1+\sigma_m}{E_m} - \frac{1+\sigma_2}{E_2}\right\}\left\{1-\rho_m^{4(2-m)}\right\}$$
$$+ \frac{C_m}{8E_m}\left\{\frac{E_m}{G_m} - 2\sigma_m\right\} \qquad [54]$$

$$\lambda_m = 1 + \rho_m^{2(2-m)} + \frac{G_m}{G_2}\left\{1 - \rho_m^{2(2-m)}\right\} \qquad [55]$$

$$\Omega_m = 1 + \rho_m^{2(2-m)} - \frac{G_m}{G_2}\left\{1 - \rho_m^{2(2-m)}\right\} . \qquad [56]$$

Conditions [10] [13], [14] and [15] have not been imposed in the determination of the components of stress and displacement. Among these [10] is implied by [3], [1] and by [16] applied to the components τ_{zrm}, while the remaining conditions, [13], [14], and [15], are satisfied identically by the forms [19], [23] and [24].

The constants appearing in expressions [31], [32] and [33] depend upon the method of fixing the supported end of the cylinder for their final determination. They are interrelated by the following expressions:

$$\alpha_1 = \alpha_2 - 2\ell r_1^2\left\{A_2\frac{(1+\sigma_2)}{E_2} - A_1\frac{(1+\sigma_1)}{E_1}\right\} \qquad [57]$$

$$\alpha_3 = \alpha_2 - 2\ell r_2^2\left\{\frac{A_2(1+\sigma_2)}{E_2} - A_3\frac{(1+\sigma_3)}{E_3}\right\} \qquad [58]$$

$$\beta_1 = \beta_2 + 2r_1^2\left\{\frac{A_2(1+\sigma_2)}{E_2} - \frac{A_1(1+\sigma_1)}{E_1}\right\} \qquad [59]$$

$$\beta_3 = \beta_2 + 2r_2{}^2 \left\{ \frac{A_2(1 + \sigma_2)}{E_2} - \frac{A_3(1 + \sigma_3)}{E_3} \right\} \qquad [60]$$

$$\gamma_1 = \gamma_2 = \gamma_3 \qquad [61]$$

$$\delta_1 = \delta_2 = \delta_3 \qquad [62]$$

$$\epsilon_1 = \epsilon_2 = \epsilon_3 \qquad [63]$$

$$\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3 \qquad [64]$$

3. <u>RESULTS FOR ISOTROPIC INNER AND OUTER LAYERS WITH CYLINDRICALLY AEOLOTROPIC INTERMEDIATE LAYER.</u>

For the isotropic layers, 1 and 3, the stress-strain relations are given by [18], and expressions [19] to [33], inclusive, m = 1, 3, are adopted as the forms of the components of stress and displacement.

The stress-strain relations and the expressions for the components of stress and displacement for the intermediate layer, 2, are taken from Reference 1. Since the notation for the elastic coefficients differs in form from those for an isotropic material, the subscript 2 is not used with these coefficients and the stress-strain relations are written

$$\left.\begin{aligned}
e_{zz2} &= \frac{1}{E_z}\tau_{zz2} - \frac{\sigma_{rz}}{E_r}\tau_{rr2} - \frac{\sigma_{\theta z}}{E_\theta}\tau_{\theta\theta 2}\\
e_{rr2} &= -\frac{\sigma_{zr}}{E_z}\tau_{zz2} + \frac{1}{E_r}\tau_{rr2} - \frac{\sigma_{\theta r}}{E_\theta}\tau_{\theta\theta 2}\\
e_{\theta\theta 2} &= \frac{\sigma_{z\theta}}{E_z}\tau_{zz2} - \frac{\sigma_{r\theta}}{E_r}\tau_{rr2} + \frac{1}{E_\theta}\tau_{\theta\theta 2}\\
e_{\theta z2} &= \frac{1}{2G_{r\theta}}\tau_{r\theta 2}\\
e_{\theta z2} &= \frac{1}{2G_{\theta z}}\tau_{\theta z2}\\
e_{zr2} &= \frac{1}{2G_{zr}}\tau_{zr2}.
\end{aligned}\right\} \qquad [65]$$

The forms of the components of stress are taken as follows:

$$\tau_{zz2} = \sum_n S_n F_n r^n (\ell - z)\cos\theta \qquad [66]$$

$$\tau_{rr2} = \sum_n S_n (n + 1) r^n (\ell - z) \cos \theta \qquad [67]$$

$$\tau_{\theta\theta 2} = \sum_n S_n (n + 1)(n + 2) r^n (\ell - z) \cos \theta \qquad [68]$$

$$\tau_{r\theta 2} = \sum_n S_n (n + 1) r^n (\ell - z) \sin \theta \qquad [69]$$

$$\tau_{\theta z2} = \left\{ -kT_k r^{k-1} + kT_{-k} r^{-k-1} - \sum_n \frac{S_n \left[k^2 F_n + 2G_{\theta z}(n + 2) H_n \right] r^{n+1}}{(n + 2)^2 - k^2} \right\} \sin \theta + G_{\theta z} D_2 r \qquad [70]$$

$$\tau_{zr} = \left\{ T_k r^{k-1} + T_{-k} r^{-k-1} + \sum_n \frac{S_n \left[(n + 2) F_n + 2G_{\theta z} H_n \right] r^{n+1}}{(n + 2)^2 - k^2} \right\} \cos \theta \qquad [71]$$

In these expressions the symbols S_n, for $n = 1$, i and j, T_k, T_{-k}, and D_2 are constants that are to be determined by the boundary and end conditions. The remaining symbols, together with i, j, and k, are defined under the List of Abbreviations.

The components of displacement for the intermediate layer are each separated into two parts as in Formulas [25] [26] , and [27] . In these formulas the starred components are given by expressions [31] , [32] , and [33] with $m = 2$. The primed components are given as follows:

$$u'_{z2} = \left[\frac{-S_1K_{11}(\ell - z)^2}{2} r + \sum_n \frac{S_nK_{2n}r^{n+2}}{(n+1)(n+2)} \right.$$

$$\left. + 2a_{55}k \left\{ T_k r^k - T_{-k} r^{-k} + k \sum_n \frac{S_n \left[(n+2)F_n + 2G_{\theta z}H_n\right] r^{n+2}}{(n+2)\left[(n+2)^2 - k^2\right]} \right\} \right] \cos\theta \qquad [72]$$

$$u'_{r2} = \left[\frac{-S_1K_{11}(\ell - z)^3}{6} + \sum_n \frac{S_nK_{2n}\, r^{n+1}(\ell - z)}{n+1} \right] \cos\theta \qquad [73]$$

$$u'_{\theta 2} = \left[\frac{S_1K_{11}(\ell - z)^3}{6} + \sum_n S_n \left[K_{3n} + \frac{(n+1)}{G_{r\theta}} \right] r^{n+1} (\ell - z) \right. \\ \left. + D_2 rz. \right] \sin\theta \qquad [74]$$

The constants appearing in the expressions for the components of stress and displacement for the three layers are determined as follows by the use of [11], [12], [16], and [17]. Those directly associated with the bending of the inner and outer layers are related to the coefficient S_n for the intermediate layer by the equations

$$A_1I_1 = \frac{\pi}{8} \sum_n S_n (n+1) r_1^{n+3} \qquad [75]$$

$$A_3I_3 = -\frac{\pi}{8} \sum_n S_n (n+1) r_2^{n+3} \qquad [76]$$

$$B_1I_1 = \frac{\pi}{8} \sum_n S_n (n+1) r_1^{n+3}\, r_o^4 \qquad [77]$$

$$B_3I_3 = -\frac{\pi}{8} \sum_n S_n (n+1) r_2^{n+3}\, r_3^4 \qquad [78]$$

$$\frac{C_1}{E_1} = \frac{C_3}{E_3} = S_1K_{11} \qquad [79]$$

With the use of these relations the constants S_n, for n = 1 and i and j, are evaluated as the solution of the following system of equations:

$$8(\sigma_1 A_1 I_1 + \sigma_3 A_3 I_3) + C_1 I_1 + C_3 I_3 + \pi \sum_n \frac{S_n F_n (r_2^{n+3} - r_1^{n+3})}{n+3} = -P. \qquad [80]$$

$$\frac{2A_1}{E_1}(1+\sigma_1)\left\{(3-4\sigma_1)r_1^4 - r_o^4\right\} - \frac{\sigma_1 C_1 r_1^4}{E_1}$$

$$= \sum_n \frac{S_n}{n+1}\left\{K_{2n} + K_{3n} + \frac{n+1}{G_{r\theta}}\right\} r_1^{n+3} \qquad [81]$$

$$\frac{2A_3}{E_3}(1+\sigma_3)\left\{(3-4\sigma_3)r_2^4 + r_3^4\right\} - \frac{\sigma_3 C_3 r_2^4}{E_3}$$

$$= \sum_n \frac{S_n}{n+1}\left\{K_{2n} + K_{3n} + \frac{n+1}{G_{r\theta}}\right\} r_2^{n+3}. \qquad [82]$$

Also

$$D_1 = D_2 = D_3 = \frac{M}{2\left[G_1 I_1 + G_3 I_3 + G_{\theta z} I_2\right]} \qquad [83]$$

The remaining constants in the expression for the components of stress and displacement, exclusive of those that determine the rigid body displacements, are given in terms of the preceding constants, as follows:

$$T_k = \frac{T_1 \lambda_3 r_1^2 r_2^{-k-1} - T_3 \lambda_1 r_1^{-k-1} r_2^2}{\Omega_1 \lambda_3 r_1^{k-1} r_2^{-k-1} - \Omega_3 \lambda_1 r_1^{-k-1} r_2^{k-1}} \qquad [84]$$

$$T_{-k} = \frac{\tau_3 \Omega_1 r_1^{k-1} r_2^{2} - \tau_3 \Omega_1 r_1^{2} r_2^{k-1}}{\Omega_1 \lambda_3 r_1^{k-1} r_2^{-k-1} - \Omega_3 \lambda_1 r_1^{-k-1} r_2^{k-1}} \tag{85}$$

where
$$\Omega_m = 1 + \rho_m^{2(2-m)} - \frac{kG_m}{G_{\theta z}}\left\{1 - \rho_m^{2(2-m)}\right\} \tag{86}$$

$$\lambda_m = 1 + \rho_m^{2(2-m)} + \frac{kG_m}{G_{\theta z}}\left\{1 - \rho_m^{2(2-m)}\right\} \tag{87}$$

and
$$\begin{aligned}
\tau_m = {} & N_m\left\{1 - \rho_m^{2(2-m)}\right\}\left\{1 + 2\rho_m^{2(2-m)}\right\} \\
& -G_m\left\{\frac{A_m}{E_m}(1 + \rho_m)\left(3 + 2\sigma_m + \frac{\sigma_m E_m}{G_m} - \rho_m^{4(2-m)}\right)\right. \\
& \left. - \frac{C_m}{8E_m}\left(2\sigma_m - \frac{E_m}{G_m}\right)\right\}\left\{1 - \rho_m^{2(2-m)}\right\} \\
& - \sum_n \frac{S_n\left\{(n+2)F_n + 2G_{\theta z}H_n\right\}}{(n+2)^2 - k^2}\left\{1 + \rho_n^{2(2-m)} - \frac{G_m}{(n+2)G_{zr}}\left(1 + \rho_m^{2(2-m)}\right)\right\} \\
& - G_m \sum_n \frac{S_n}{2(n+1)(n+2)}\left\{nK_{2n} - (n+2)\left(K_{3n}\frac{-(n+1)(n+2)}{G_{r\theta}}\right.\right\} r_1^{n-1}
\end{aligned} \tag{88}$$

$$\begin{aligned}
J_1 = {} & \frac{1}{r_1^2 - r_1^2}\left\{T_r r_1^{k+1} + T_{-k} r_1^{-k+1} + \sum_n \frac{S_n\left\{(n+2)F_n + 2G_{\theta z}H_n\right\} r_1^{n+3}}{(n+2)^2 - k^2}\right\} \\
& -N_1\left(r_1^2 + r_0^2\right).
\end{aligned} \tag{89}$$

$$L_1 = -J_1 r_0^2 - N_1 r_0^4. \tag{90}$$

$$J_3 = \frac{1}{r_2^2 - r_3^2}\left\{T_k r_2^{k+1} + T_{-k} r_2^{-k-1} + \sum_n \frac{S_n\left\{(n+2)F_n + 2G_{\theta z} H_n\right\} r_2^{n+3}}{(n+2)^2 - k^2}\right\}$$

$$-N_3(r_2^2 + r_3^2 \qquad [91]$$

and

$$L_3 = -J_3 r_3^2 - N_3 r_3^4. \qquad [92]$$

Conditions [10] , [13] , [14] and [15] are again satisfied for reasons enumerated in Section 2.

The constants that determine the rigid body displacements are interrelated by the following equations:

$$\alpha_1 - \frac{2A_1}{E_1}(1 + \sigma_1)\ell r_1^2 = \alpha_2 + \sum_n \frac{S_n}{2(n+1)}\left\{K_{2n} - K_{3n} - \frac{n+1}{G_{r\theta}}\right\} r_1^{n+1}\ell \qquad [93]$$

$$\alpha_3 - \frac{2A_3}{E_3}(1 + \sigma_3)\ell r_2^2 = \alpha_2 + \sum_n \frac{S_n}{2(n+1)}\left\{K_{2n} - K_{3n} - \frac{n+1}{G_{r\theta}}\right\} r_2^{n+1}\ell \qquad [94]$$

$$\beta_1 + \frac{2A_1}{E_1}(1 + \sigma_1)r_1^2 = \beta_2 - \sum_n \frac{S_n}{2(n+1)}\left\{K_{2n} - K_{3n} - \frac{n+1}{G_{r\theta}}\right\} r_1^{n+1} \qquad [95]$$

$$\beta_3 + \frac{2A_3}{E_3}(1 + \sigma_3)r_2^2 = \beta_2 - \sum_n \frac{S_n}{2(n+1)}\left\{K_{2n} - K_{3n} - \frac{n+1}{G_{r\theta}}\right\} r_2^{n+1} \qquad [96]$$

$$\gamma_1 = \gamma_2 = \gamma_3 \qquad [97]$$

$$\delta_1 = \delta_2 = \delta_3 \qquad [98]$$

$$\epsilon_1 = \epsilon_2 = \epsilon_3 \qquad [99]$$

$$\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 \qquad [100]$$

4. RESULTS FOR EQUAL POISSON'S RATIOS.

When the Poisson's ratios of the three layers are equal, the expressions for the components of stress and displacement are considerably simplified. These simplified results are given in this section both for the case of an isotropic intermediate layer and that of a cylindrically aeolotropic intermediate layer.

Since the effects of varying the Poisson's ratios are expected to be small, at least within the ranges of these ratios found in commonly used engineering materials, the formulas of the present section are considered to be applicable for obtaining approximate results for the general cases of the two preceding sections.

Case A, Cylinder with an Isotropic Intermediate Layer with

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma.$$

When the Poisson's ratios are equal formulas [34] through [39] reduce to

$$A_1 = A_2 = A_3 = 0, \qquad [101]$$

and

$$B_1 = B_2 = B_3 = 0. \qquad [102]$$

Also, from [40] and [43],

$$\frac{C_1}{E_1} = \frac{C_2}{E_2} = \frac{C_3}{E_3} = \frac{-P}{EI}. \qquad [103]$$

where

$$EI = E_1 I_1 + E_2 I_2 + E_3 I_3. \qquad [104]$$

The components of stress with D_m from [41], then reduce to the forms

$$\tau_{zzm} = \frac{-E_m P}{EI} r (\ell - z) \cos\theta. \qquad [105]$$

$$\tau_{rrm} = \tau_{\theta\theta m} = \tau_{r\theta m} = 0. \qquad [106]$$

$$\tau_{zrm} = \frac{-P}{EI}\left[J_m' + \frac{L_m'}{r^2} + \frac{r^2}{8}(3E_m - 2\sigma G_m)\right] \cos\theta, \qquad [107]$$

and
$$\tau_{\theta zm} = \frac{P}{EI}\left[J_m' - \frac{L_m'}{r^2} + \frac{r^2}{8}(E_m - 6\sigma G_m)\right] \sin\theta + \frac{MG_m r}{2GI} \qquad [108]$$

where

$$GI = G_1 I_1 + G_2 I_2 + G_3 I_3 \qquad [109]$$

and where

$$J_m' = \frac{J_m EI}{P} \qquad [110]$$

$$L_m' = \frac{L_m EI}{P}. \qquad [111]$$

The constants J_m and L_m are determined by [46] to [51], inclusive with

$$N_m = \frac{-P}{8EI}(3E_m - 2\sigma G_m), \qquad [112]$$

$$\tau_m = -\frac{2}{3} N_m \left\{ 1 - \rho_m^{2(2-m)} \right\} \left\{ 1 + 3\rho_m^{2(2-m)} \right\}$$

$$+ N_2 \left[1 + \rho_m^{2(2-m)} - \frac{G_m}{3G_2} \left\{ 1 - \rho_m^{2(2-m)} \right\} \right], \qquad [113]$$

and with λ_m and Λ_m as given by [55] and [56], respectively.

The components of displacement associated with the preceding stress components are given by the expressions

$$u'_{zm} = \frac{P}{EI} \left[\frac{r(\ell - z)^2}{2} + \frac{r^3}{8} \left(2\sigma - \frac{E_m}{G_m} \right) \right.$$

$$\left. - \frac{r}{G_m} \left(J'_m - \frac{L'_m}{r^2} \right) \right] \cos\theta \qquad [114]$$

$$u'_{rm} = \frac{P}{EI} \left[\frac{(\ell - z)^3}{6} + \frac{\sigma}{2} r^2 (\ell - z) \right] \cos\theta \qquad [115]$$

$$u'_{\theta m} = \frac{P}{EI} \left[- \frac{(\ell - z)^3}{6} + \frac{\sigma}{2} r^2 (\ell - z) \right] \sin\theta. \qquad [116]$$

Formulas [31], [32] and [33] for the rigid body displacements remain as given.

As an example of the manner in which the rigid body displacements may be fixed, consider the conditions

$$\left. \begin{aligned} u_z &= 0 \quad \text{for } z = 0, \; r = r_1, \; 0 \le \theta \le 2\pi \\ u_r &= 0 \quad \text{for } z = 0, \; r = r_1, \; \theta = \pm \frac{\pi}{2} \\ u_\theta &= 0 \quad \text{for } z = 0, \; r = r_1, \; \theta = \pm \frac{\pi}{2} \end{aligned} \right\} \qquad [117]$$

These conditions fix the plane of the circle $z = 0$, $r = r_1$ and the two points on the circle at $\theta = \pm \frac{\pi}{2}$. With these conditions, the constants in expressions [31], [32] and [33] take on the values

$$\alpha_m = \frac{-P}{EI} \left\{ \frac{\ell^3}{6} - \frac{\sigma r_1^2}{2} \right\} \tag{118}$$

$$\beta_m = \frac{P}{EI} \left\{ \frac{\ell^2}{2} + \frac{r_1^2}{8} \left(2\sigma - \frac{E_2}{G_2} \right) - \frac{1}{G_2} \left(J_2' - \frac{L_2'}{r_1^2} \right) \right\} \tag{119}$$

$$\gamma_m = \delta_m = \delta_m = \epsilon_m = 0 \tag{120}$$

It is observed that expressions [115], [116], [118], [119] and [120] are independent of the index m. The components u_r and u_θ for a given layer do not then depend in a special way upon the elastic coefficients for the material of the layer but rather upon the elastic properties of the composite cylinder. Therefore, the deflection of the cylinder,

$$u = u_r \cos\theta - u_\theta \sin\theta \tag{121}$$

is given by the formula

$$u = \frac{P}{EI} \left[\frac{(\ell - z)^3}{6} + \frac{\sigma}{2} (\ell - z)\, r^2 \cos 2\theta \right] + \alpha_m + \beta_m Z - \frac{Mzr \sin\theta}{2GI} \tag{122}$$

with α_m and β_m given by [118] and [119], respectively. With the use of the latter formulas the deflection at the end $z = \ell$ is obtained in the form

$$u\Big|_{z=\ell} = \frac{P}{EI}\left[\frac{\ell^3}{3} + \frac{r_1^2\ell}{8}\left(6\sigma - \frac{E_2}{G_2}\right)\right.$$

$$\left.\frac{\ell}{G_2}\left\{\frac{r_1^2\,T_1'\,(r_1^2\lambda_3 + r_2^2\Omega_3) - r_2^4\,T_3'\,(\lambda_1 + \Omega_1)}{r_1^2\Omega_1\lambda_3 - r_2^2\Omega_3\lambda_1}\right\}\right]$$

$$-\frac{M\ell}{2GI}\, r \sin\theta \qquad [123]$$

where $$T_m' = \frac{T_m EI}{P} \qquad [124]$$

with T_m given by [113] .

Case B, Cylindrically Aeolotropic Intermediate Layer with

$$\sigma_1 = \sigma_3 = \sigma_{zr} = \sigma_{z\theta} = \sigma \text{ [4]}$$

When $\sigma_{zr} = \sigma_{z\theta}$ the quantities F_1 and H_1 are undefined. However, the results of Section 3 are applicable under the present conditions with

$$S_1 = S_2 = S_3 = 0 \qquad [125]$$

and with

$$S_1K_{11} = -\frac{P}{EI} \qquad [126]$$

and

$$S_1H_1 = S_1K_{21} = S_1K_{31} = -\frac{\sigma}{E_z}S_1F_1 = \frac{\sigma P}{EI} \qquad [127]$$

where now

$$EI = E_1I_1 + E_zI_2 + E_3I_3. \qquad [128]$$

---

[4] The remaining Poisson's ratios of the intermediate layer are not assumed to be equal.

With the substitution of [125] and [126] into [75] to [79] inclusive, it is found that

$$A_1 = A_3 = B_1 = B_3 = 0 \qquad [129]$$

and

$$\frac{C_1}{E_1} = \frac{C_2}{E_2} = -\frac{P}{EI}\,. \qquad [130]$$

The components of stress for layers 1 and 3 are of the forms [105] to [108] inclusive, with J_m' and L_m' defined below and, since the constants D_m are now determined by [83], with

$$GI = G_1 I_1 + G_{\theta z} I_2 + G_3 I_3. \qquad [131]$$

For the intermediate layer,

$$\tau_{zz2} = -\frac{E_z P}{EI}\, r(\ell - z)\cos\theta \qquad [132]$$

$$\tau_{rr2} = \tau_{\theta\theta 2} = \tau_{r\theta 2} = 0 \qquad [133]$$

$$\tau_{zr2} = \frac{P}{EI}\left[T_k' r^{k-1} + T_{-k}' r^{-k-1} - \frac{(3E_2 - 2\sigma G_{\theta z})r^2}{9 - k^2}\right]\cos\theta \qquad [134]$$

$$\tau_{\theta z2} = \frac{-P}{EI}\left[kT_k' r^{k-1} - kT_{-k}' r^{-k-1} + \frac{(k^2 E_z - 6\sigma G_{\theta z})r^2}{9 - k^2}\right]\sin\theta + \frac{MG_{\theta z} r}{2GI} \qquad [135]$$

where

$$T_k' = \frac{T_k EI}{P} \qquad [136]$$

$$T'_{-k} = \frac{T_{-k}EI}{P}$$ [137]

with T_k and T_{-k} determined from [84] and [85], respectively. For the determinations of these two constants

$$\mathcal{T}_m = \frac{-(3E_m - 2\sigma G_m)}{12}\left\{1 - \rho_m^{2(2-m)}\right\}\left\{1 + 3\rho_m^{2(2-m)}\right\}$$

$$+ \frac{(3E_z - 2\sigma G_{\theta z})}{9 - k^2}\left[1 + \rho_m^{2(2-m)} - \frac{G_m}{3G_{zr}}\left\{1 - \rho_m^{2(2-m)}\right\}\right]$$ [138]

and the quantities Ω_m and λ_m remain as given by [86] and [87], respectively. Having evaluated T'_k and T'_{-k}, the constants J'_m and L'_m are determined from the equations

$$J'_1 = \frac{1}{r_1^2 - r_0}\left\{T'_k r_1^{k+1} + T'_{-k} r_1^{-k+1} - \frac{(3E_z - 2\sigma G_{\theta z})}{9 - k^2} r_1^4\right\} - N'_1(r_1^2 + r_2^2)$$

$$L'_1 = -J'_1 r_0^2 - N'_1 r_0^4$$

$$J'_3 = \frac{1}{r_2^2 - r_3^2}\left\{T'_k r_2^{k+1} + T'_{-k} r_2^{-k+1} - \frac{(3E_z - 2\sigma G_{\theta z})}{9 - k^2} r_2^4\right\}$$

$$- N'_3 (r_2^2 + r_3^2)$$

$$L'_3 = -J'_3 r_3^2 - N'_3 r_3^4$$ [139]

where

$$N'_m = \frac{1}{8}(3E_m - 2\sigma G_m).$$ [140]

The components of displacement for layers 1 and 3 are obtained from [31], [32], [33], [114], [115] and [116]. Those for the intermediate layer are obtained from [31] to [33] inclusive and the expressions

$$u'_{z2} = \frac{P}{EI}\left[\frac{(\ell - z)^2 r}{2} + \left\{\frac{\sigma}{6} - \frac{3E_z - 2\sigma G_{\theta z}}{3(9 - k^2)G_{zr}}\right\} r^3 + \frac{k}{G_{\theta z}}\left\{T'_k r^k - T'_{-k} r^{-k}\right\}\right]\cos\theta \qquad [141]$$

$$u'_{r2} = \frac{P}{EI}\left[\frac{(\ell - z)^3}{6} + \frac{\sigma r^2}{2}(\ell - z)\right]\cos\theta \qquad [142]$$

$$u'_{\theta 2} = \frac{P}{EI}\left[-\frac{(\ell - z)^3}{6} + \frac{\sigma r^2}{2}(\ell - z)\right]\sin\theta + \frac{Mrz}{2GI}. \qquad [143]$$

For the case under consideration, conditions [117] lead to the evaluations

$$\alpha_m = \frac{-P}{EI}\left[\frac{\ell^3}{6} - \frac{\sigma r_1^2 \ell}{2}\right] \qquad [144]$$

$$\beta_m = \frac{P}{EI}\left[\frac{\ell^2}{2} + \left\{\frac{\sigma}{6} - \frac{3E_z - 2\sigma G_{\theta z}}{3(9 - k^2)G_{zr}}\right\} r_1^2 - \frac{k}{G_{\theta z}}\left\{T'_k r_1^{k-1} - T'_{-k} r_1^{-k-1}\right\}\right] \qquad [145]$$

and

$$\gamma_m = \delta_m = \delta_m = \epsilon_m = 0. \qquad [146]$$

The deflection of the cylinder is obtained from [122] with the use of [144] and [145]. At $z = \ell$ the deflection is given by

$$u\Big|_{z=\ell} = \frac{P}{EI}\left[\frac{\ell^3}{3} + \frac{\ell r_1^2}{3}\left\{2\sigma - \frac{3E_z - 2\sigma G_{\theta z}}{(9-k^2)G_{zr}}\right\} + \frac{k\ell\left\{T_1'(\lambda_3 r_1^{k+1} r_2^{-k-1} + \Omega_3 r_1^{-k+1} r_2^{k-1}) - T_3' r_1^{-2} r_2^{2}(\lambda_1 + \Omega_1)\right\}}{G_{\theta z}(\Omega_1 \lambda_3 r_1^{k-1} r_2^{-k-1} - \Omega_3 \lambda_1 r_1^{-k-1} r_2^{k-1})}\right]$$

$$- \frac{M\ell r \sin\theta}{2GI} \qquad [147]$$

where

$$T_m' = \frac{T_m EI}{P} \qquad [148]$$

with T_m defined by [138] and EI by [128] .

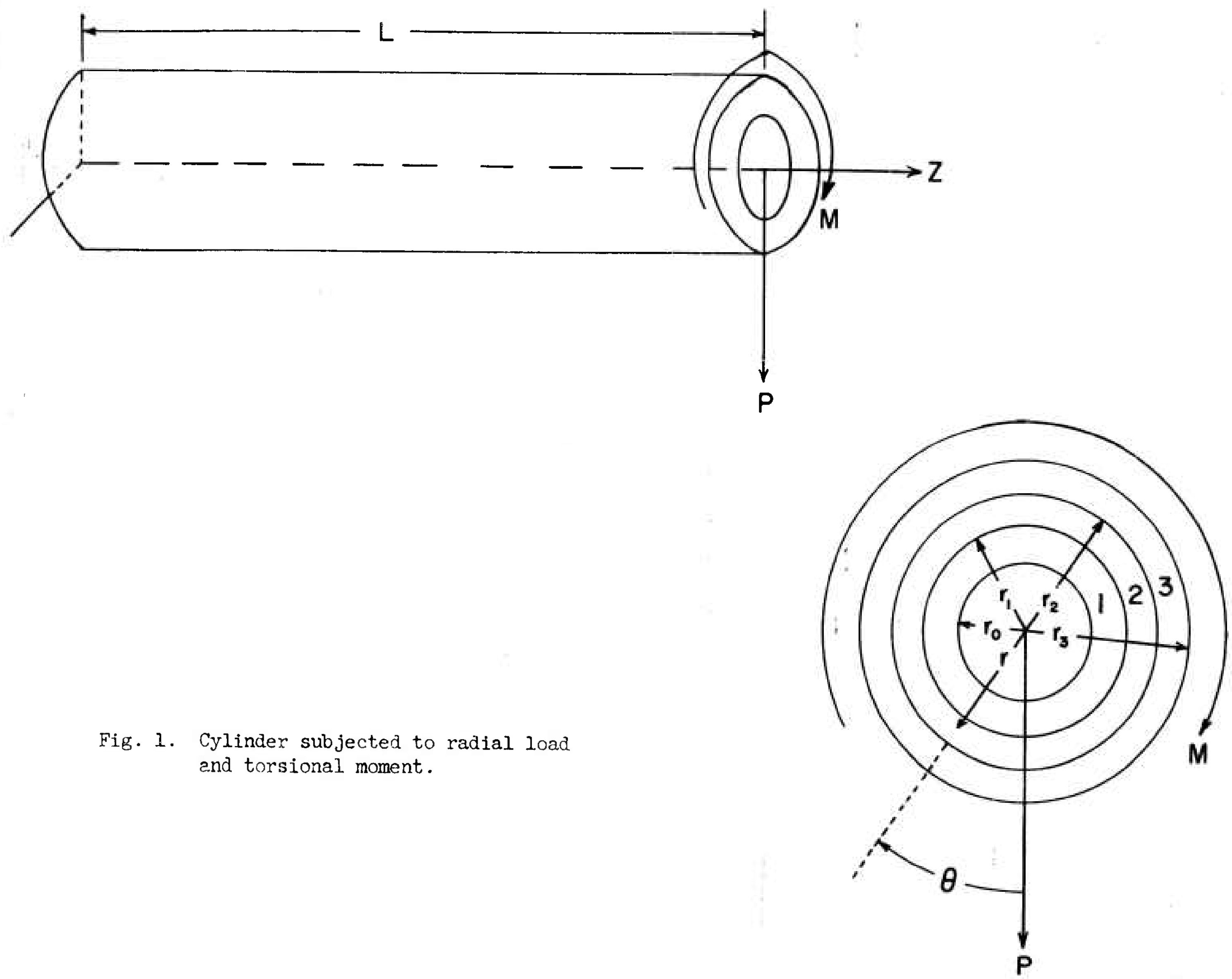

Fig. 1. Cylinder subjected to radial load and torsional moment.

SECTION IV

BUCKLING OF SANDWICH CYLINDERS IN TORSION[1,2]

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Summary

This report presents a mathematical analysis leading to the critical stresses that determine the buckling of cylinders of sandwich construction in torsion. The analysis is complete for cylinders of finite length having orthotropic facings and orthotropic cores. Formulas are given for sandwich cylinders of finite and infinite length. Curves are given for buckling coefficients for cylinders with isotropic facings and orthotropic cores, over a wide range of cylinder sizes and material properties.

Introduction

The purpose of this report is to develop formulas by means of which the buckling stress can be calculated for a sandwich cylinder subjected to torsion.

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[1] This report is one of a series (ANC-23, Item 57-3) originally prepared and distributed in June 1953 by the Forest Products Laboratory under U. S. Navy Bureau of Aeronautics Order No. NAer 01336 and U. S. Air Force No. USAF-33(038)-51-4326-E, Amend. 2(53-131). This revision was prepared under U. S. Navy Bureau of Aeronautics No. 01778 and 01768 and U. S. Air Force No. 33(616)56-9, Amend. A3(57-398). Further revision may be made as additional data become available.

[2] Revised December 1957 by Charles B. Norris, Engineer.

[3] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Methods have previously been developed to determine the buckling stress of homogeneous isotropic cylinders (1)[4] and of plywood cylinders (7). The analysis for plywood cylinders (7) based on an energy method, will be used in the present report, with suitable modifications to take into account the effect of shear deformation in the core of the sandwich shell.

The core and facings of the cylindrical sandwich shell will be taken to be orthotropic with two of their natural axes parallel to the axial and circumferential directions, respectively.

The facings, which may be of either equal or unequal thickness, are assumed to be thin, but their flexural rigidities are not neglected, as they may be of importance in certain cases. All stress components in the core are neglected except the transverse shear components, and these are taken to be constant across the thickness of the core. An energy method will be employed to determine the buckling stress. In contrast with the treatment of sandwich cylinders in axial compression (6) in which a large-deflection theory was used, the displacements will here be assumed to be such that a small-deflection theory is applicable. The use of a small-deflection theory instead of a large-deflection theory constitutes the chief difference in the treatment of the two problems. The consequence is that in applying an energy method a large part of the analysis for both problems is the same in its main features. In order to avoid repetition of the details of exactly the same steps in the present report that are to be found in the previous report on sandwich cylinders in axial compression (6), reference will frequently be made to that report. Sufficient details will be presented here to show the principal steps in the analysis and the differences that result from using a small-deflection theory and a different form of the buckled surface from that chosen for cylinders in axial compression.

Large-deflection theories for cylindrical shells subjected to torsion will yield only slightly lower critical shear stresses than small-deflection theories if the imperfections in the cylindrical surface are small compared to the thickness of the shell (8, 12). Cylinders of sandwich construction usually exhibit exceedingly small imperfections; thus, the small-deflection theory given here seems adequate.

Choice of Axes and Notation

The choice of axes is shown in figure 1. The coordinate y is measured along the circumference. The components of displacement are u, v, and w, respectively, where w is positive inward. Love's notation (4) will be used for

[4] Underlined numbers in parentheses refer to numbered references under Literature Cited at end of report.

the components of strain e_{xx}, e_{xy}, etc., and for the components of stress X_x, X_y, etc. The thickness of the core will be denoted by c and that of each facing by f_1, and f_2, respectively, as in figure 2.

Form of the Buckled Surface

As an energy method is to be used to determine the buckling stress, it is necessary to assume a suitable form for the buckled surface. The following form will be chosen:

$$\frac{w}{r} = g \sin \alpha(y - \gamma x) \sin \beta x \tag{1}$$

where

$$\alpha = \frac{n}{r}, \quad \beta = \frac{\pi}{b} \tag{2}$$

n = the number of buckles in the circumferential direction and b = the length of the cylinder. The first trigonometric factor in equation (1) is suggested by the form that has been used to represent the buckled surface of an infinite plane strip (10) under uniform shear. The factor sin βx is introduced to make the deflection vanish at the ends of the cylindrical shell. Further conditions at the ends are disregarded. Except for very short cylinders, these conditions are not important.

It is convenient to write equation (1) of the buckled surface in the form

$$\frac{w}{r} = \frac{g}{2} \left[\cos (\alpha y - \delta x) - \cos (\alpha y - \epsilon x)\right] \tag{3}$$

where

$$\delta = \alpha\gamma + \beta, \quad \epsilon = \alpha\gamma - \beta \tag{4}$$

Extensional Strain and Stress Components

Expressions will now be written for the extensional strains uniform across the thickness of the cylindrical shell and for the corresponding mean membrane stress components. On these will be superposed a system of flexural strains, and the energy of deformation associated with each system of strains will be found.

The extensional strain components are expressed by the equations:

$$
\begin{aligned}
e_{xx} &= \frac{\partial u}{\partial x} \\
e_{yy} &= \frac{\partial v}{\partial y} - \frac{w}{r} \\
e_{xy} &= \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}
\end{aligned} \qquad (5)
$$

In each facing the corresponding stress components are:

$$
\begin{aligned}
X_x &= \frac{E_x}{\lambda} (e_{xx} + \sigma_{yx} e_{yy}) \\
Y_y &= \frac{E_y}{\lambda} (e_{yy} + \sigma_{xy} e_{xx}) \\
X_y &= \mu_{xy} e_{xy}
\end{aligned} \qquad (6)
$$

where E_x and E_y are Young's moduli, μ_{xy} is the modulus of rigidity for shearing strains referred to the x and y directions, σ_{xy} and σ_{yx} are Poisson's ratios and $\lambda = 1 - \sigma_{xy} \sigma_{yx}$. All of these quantities are elastic properties of the facings. Because the stress components X_x, Y_y, and X_y are neglected in the core, the mean membrane stress components for the shell are:

$$
\begin{aligned}
\overline{X}_x &= \frac{E_a}{\lambda} (e_{xx} + \sigma_{yx} e_{yy}) \\
\overline{Y}_y &= \frac{E_b}{\lambda} (e_{yy} + \sigma_{xy} e_{xx}) \\
\overline{X}_y &= \mu_m e_{xy}
\end{aligned} \qquad (7)
$$

where

$$E_a = E_x \left(1 - \frac{c}{h}\right), \quad E_b = E_y \left(1 - \frac{c}{h}\right), \quad \mu_m = \mu_{xy} \left(1 - \frac{c}{h}\right) \qquad (8)$$

and

$$h = c + f_1 + f_2 \qquad (9)$$

By using the relation

$$E_a \sigma_{yx} = E_b \sigma_{xy}$$

which can be obtained from the relation

$$E_x \sigma_{yx} = E_y \sigma_{xy},$$

it follows from equations (6) that:

$$e_{xx} = \frac{1}{E_a}\overline{X}_x - \frac{\sigma_{yx}}{E_b}\overline{Y}_y = \frac{1}{E_a}\overline{X}_x - \frac{\sigma_{xy}}{E_a}\overline{Y}_y$$

$$e_{yy} = \frac{1}{E_b}\overline{Y}_y - \frac{\sigma_{xy}}{E_a}\overline{X}_x = \frac{1}{E_b}\overline{Y}_y - \frac{\sigma_{yx}}{E_b}\overline{X}_x \qquad (10)$$

$$e_{xy} = \frac{1}{\mu_m}\overline{X}_y$$

The mean membrane stress components satisfy the equations of equilibrium:

$$\frac{\partial \overline{X}_x}{\partial x} + \frac{\partial \overline{X}_y}{\partial y} = 0$$

$$\qquad (11)$$

$$\frac{\partial \overline{X}_y}{\partial x} + \frac{\partial \overline{Y}_y}{\partial y} = 0$$

They can consequently be expressed in terms of a stress function as follows:

$$\overline{X}_x = \frac{\partial^2 F}{\partial y^2}, \quad \overline{Y}_y = \frac{\partial^2 F}{\partial x^2}, \quad \overline{X}_y = -\frac{\partial^2 F}{\partial x \partial y} \qquad (12)$$

By eliminating u and v from equation (5), it is found that

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - \frac{\partial^2 e_{xy}}{\partial x \partial y} = -\frac{1}{r}\frac{\partial^2 w}{\partial x^2} \qquad (13)$$

The following differential equation for the stress function F is found by substituting equations (12) in equations (10) and the results of those substitutions in equation (13):

$$A\frac{\partial^4 F}{\partial x^4} + C\frac{\partial^4 F}{\partial x^2 \partial y^2} + B\frac{\partial^4 F}{\partial y^4} = -\frac{1}{r}\frac{\partial^2 w}{\partial x^2} \qquad (14)$$

where[5]

$$A = \frac{1}{E_b}, \quad B = \frac{1}{E_a}, \quad C = \frac{1}{\mu_m} - \frac{2\sigma_{xy}}{E_a}$$

For comparison with Report No. 1529 (7) set $\theta = 0$ in that report and interchange A and B.

The expression for w from equation (3) is substituted in equation (14). A solution of the resulting equation is found to be

$$F = a_1 \cos(\alpha y - \delta x) + a_2 \cos(\alpha y - \epsilon x) \tag{15}$$

where

$$a_1 = \frac{g\delta^2}{2K_1}, \quad a_2 = -\frac{g\epsilon^2}{2K_2} \tag{16}$$

$$K_1 = A\delta^4 + C\alpha^2\delta^2 + B\alpha^4 \tag{17}$$

$$K_2 = A\epsilon^4 + C\alpha^2\epsilon^2 + B\alpha^4 \tag{18}$$

In accordance with equations (12) the stress components can be found from the stress function F.

Extensional Strain Energy

The strain energy W_1 associated with the membrane strains (5) is expressed by the integral

$$W_1 = \frac{h}{2} \int_0^b \int_0^{2\pi r} (\bar{X}_x e_{xx} + \bar{Y}_y e_{yy} + \bar{X}_y e_{xy})\, dydx \tag{19}$$

With the aid of equations (10), this integral becomes

---

[5] The definitions of A and B of reference (7) are interchanged here to agree with the definitions used in reference (6). It can be shown that H of reference (7) is replaced here by E_aE_b.

$$W_1 = \frac{h}{2}\int_0^b \int_0^{2\pi r} \left(B\overline{X}_x^{\,2} + A\overline{Y}_y^{\,2} - \frac{2\sigma_{xy}}{E_a}\overline{X}_x\overline{Y}_y + \frac{\overline{X}_y^{\,2}}{\mu_m}\right) dydx \qquad (20)$$

By substituting the values of $\overline{X}_x$, $\overline{Y}_y$, and $\overline{X}_y$ that are obtained from equations (12) and (15) and performing the integrations, it is found that

$$W_1 = \frac{\pi rbh}{2}\left[B(a_1^2 + a_2^2)\,\alpha^4 + A(a_1^2\delta^4 + a_2^2\epsilon^4) + C(a_1^2\delta^2 + a_2^2\epsilon^2)\,\alpha^2\right] \qquad (21)$$

By using equations (16), (17), and (18), the expression for $\underline{W_1}$ is reduced to the simpler form:

$$W_1 = \frac{\pi rbhg^2}{8}\left[\frac{\delta^4}{K_1} + \frac{\epsilon^4}{K_2}\right] \qquad (22)$$

Flexural Energy of the Shell

As in reference (6), the following expressions will be used for the curvatures and unit twist:

$$\frac{\partial^2 w}{\partial x^2}, \quad \frac{\partial^2 w}{\partial y^2}, \quad \frac{\partial^2 w}{\partial x \partial y},$$

These expressions are exactly those used in calculating the flexural energy of a flat sandwich plate. The approximate flexural energy of such a plate was found in references (5) and (2) by using the tilting method of Williams, Leggett, and Hopkins (11). In this method the transverse shear components are taken to be constant across the thickness of the core. The following expressions are chosen for the components of the displacement in the core:

$$u_c = -k\,(\zeta - q)\frac{\partial w}{\partial x}$$

$$v_c = -k'\,(\zeta - q')\frac{\partial w}{\partial y} \qquad (23)$$

$$w_c = w\,(x,\ y)$$

The coordinate ζ and the distance q are shown in figure 2. Thus $\zeta = q$ denotes the surface in which the components of displacement in the x direction vanish and the parameter k describes the inclination in the x direction to the normal of the deformed surface of the panel, of lines that were normal to the undeformed surface before deformation took place. Similarly q' and k' are related to the displacements in the y direction.

In obtaining the expression for the flexural energy of deformation, which will be denoted by W_2, it is assumed that all stress components in the core may be neglected except the transverse shear components. The facings are treated as thin plates.

The details of the calculation of W_2 and its minimum value will be found in reference (6), which in turn depends for some of its details on reference (2). The following expression is found for the minimum value of W_2 with respect to the quantities k, k', q, and q':

$$W_2 = \frac{\pi b r^3 g^2}{8} \left\{ \frac{I\left[A_1' + 2A_2' + A_3' + (A_1'A_3' - A_2'^2)\left(\frac{\phi}{A_4'} + \frac{\phi}{A_5'}\right)\right]}{1 + \frac{A_1'\phi}{A_4'} + \frac{A_3'\phi}{A_5'} + \frac{\phi^2 (A_1'A_3' - A_2'^2)}{A_4'A_5'}} + I_f (A_1' + 2A_2' + A_3') \right\} \tag{24}$$

where the coefficient of the expression in brackets is different from that in reference (6) because the derivatives of a different function w appear in the integrals to be evaluated, and where

$$I = \frac{f_1 f_2}{4(f_1 + f_2)} (h + c)^2 \tag{25}$$

$$I_f = \frac{f_1^3 + f_2^3}{12} \tag{26}$$

$$\phi = \frac{c f_1 f_2}{f_1 + f_2} \tag{27}$$

$$H = A_1' + 2A_2' + A_3' \tag{28}$$

$$A_1' = \frac{1}{\lambda}\left[E_x (\delta^4 + \epsilon^4) + \lambda \mu_{xy} \alpha^2 (\delta^2 + \epsilon^2)\right] \tag{29}$$

$$A_2' = \frac{1}{\lambda}\left[E_x\,\sigma_{yx} + \lambda\mu_{xy}\right]\alpha^2\,(\delta^2 + \epsilon^2) \tag{30}$$

$$A_3' = \frac{1}{\lambda}\left[2E_y\,\alpha^4 + \lambda\mu_{xy}\,\alpha^2\,(\delta^2 + \epsilon^2)\right] \tag{31}$$

$$A_4' = \mu'_{\zeta x}\,(\delta^2 + \epsilon^2) \tag{32}$$

$$A_5' = 2\mu'_{y\zeta}\,\alpha^2 \tag{33}$$

In equations (32) and (33) $\mu'_{\zeta x}$ and $\mu'_{y\zeta}$ denote the transverse moduli of rigidity of the core. The elastic properties of the facings are denoted by unprimed letters in equations (29), (30), and (31).

It follows from equations (28), (29), (30), and (31) that

$$H = \frac{1}{\lambda}\left[E_x\,(\delta^4 + \epsilon^4) + 2E_y\,\alpha^4 + 2L\,\alpha^2\,(\delta^2 + \epsilon^2)\right] \tag{34}$$

where

$$L = E_x\,\sigma_{yx} + 2\lambda\mu_{xy} \tag{35}$$

Work Done by the Applied Couple

The work W_3 done by the applied couple in producing the deformation of the cylindrical surface described by equation (1) is given by the following integral:

$$W_3 = -\,\tau h \int_0^b \int_0^{2\pi r} \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dy\,dx \tag{36}$$

where τ is the mean shearing stress induced in the cylindrical shell by the couples applied at its ends. The corresponding stress in the facings is given by:

$$\tau_f = \frac{\tau}{1 - \frac{c}{h}}$$

Substitution of $\underline{w}$ from equation (1) in equation (36) yields

$$W_3 = \frac{\tau h b \pi r^3 g^2 \alpha^2 \gamma}{2} \tag{37}$$

Critical Stress

The critical shearing stress is determined from the equation

$$W_3 = W_1 + W_2.$$

On substituting the expressions for $\underline{W_1}$, $\underline{W_2}$, and $\underline{W_3}$ from equations (22), (24), and (37), and solving the resulting equation for $\underline{\tau}$, it is found that

$$\tau = \frac{1}{4r^2\alpha^2\gamma}\left[\frac{\delta^4}{K_1} + \frac{\epsilon^4}{K_2}\right] + \frac{1}{4h\alpha^2\gamma}\left\{\frac{I\left[A_1' + 2A_2' + A_3' + (A_1'A_3' - A_2'^2)\left(\frac{\phi}{A_4'} + \frac{\phi}{A_5'}\right)\right]}{1 + \frac{A_1'\phi}{A_4'} + \frac{A_3'\phi}{A_5'} + \frac{\phi^2(A_1A_3 - A_2'^2)}{A_4'A_5'}} + I_f(A_1' + 2A_2' + A_3')\right\} \tag{38}$$

The mean critical shearing stress is the least value of $\underline{\tau}$ that satisfies equation (38). The justification for the previous calculation of the minimum of $\underline{W_2}$ with respect to $\underline{k}$, $\underline{k'}$, $\underline{q}$, and $\underline{q'}$ is found in the fact that these quantities occur only in the expression for $\underline{W_2}$.

It will be convenient to introduce coefficients $\underline{K}$ and $\underline{K_f}$ by the equations

$$\tau = KE_x \frac{h}{r} \quad \text{and} \quad \tau_f = K_f E_x \frac{h}{r} \tag{39}$$

where $\underline{\tau_f}$ is the shear stress in the facings at the critical load and

$$K_f = \frac{K}{1 - \frac{c}{h}}$$

Then

$$K = \frac{1}{4E_x rh\alpha^2\gamma}\left[\frac{\delta^4}{K_1} + \frac{\epsilon^4}{K_2}\right] + \frac{r}{4h^2\alpha^2\gamma E_x}\left\{\frac{I\left[A_1' + 2A_2' + A_3' + (A_1'A_3' - A_2'^2)\left(\frac{\phi}{A_4'} + \frac{\phi}{A_5'}\right)\right]}{1 + \frac{A_1'\phi}{A_4'} + \frac{A_3'\phi}{A_5'} + \frac{\phi^2(A_1'A_3' - A_2'^2)}{A_4'A_5'}} + I_f(A_1' + 2A_2' + A_3')\right\} \quad (40)$$

Equation (40) can be put into a convenient form by a change of parameters. Consider the first term and place

$$\rho = \frac{\beta}{\alpha} = \frac{\pi r}{nb} \quad (41)$$

and

$$J = \frac{b^2}{hr} \quad (42)$$

Then in accordance with equations (4)

$$\delta = \alpha(\gamma + \rho), \qquad \epsilon = \alpha(\gamma - \rho) \quad (43)$$

Substituting the expressions given by equations (8), (41), (42), (43), and the relations given after equation (14) into the first term of equation (40) and dividing by $1 - \frac{c}{h}$, the following expression is obtained:

$$\psi_1 = \frac{J\rho^2}{4\pi^2\gamma}\left[\frac{(\gamma+\rho)^4}{\frac{E_x}{E_y}(\gamma+\rho)^4 + (\frac{E_x}{\mu_{xy}} - 2\sigma_{xy})(\gamma+\rho)^2 + 1}\right.$$

$$\left. + \frac{(\gamma-\rho)^4}{\frac{E_x}{E_y}(\gamma-\rho)^4 + (\frac{E_x}{\mu_{xy}} - 2\sigma_{xy})(\gamma-\rho)^2 + 1}\right] \tag{44}$$

For the second term of equation (40) the following additional parameters are convenient:

$$S_x = \frac{\phi E_x}{\lambda\mu'_{\zeta x} rh} \quad \text{and} \quad S_y = \frac{\phi E_x}{\lambda\mu'_{y\zeta} rh} \tag{45}$$

Substitution of the expression given by equations (43) into equations (29), (30), and (31) yield:

$$A_1' = \alpha^4 E_x A_1''; \quad A_2' = \alpha^4 E_x A_2''; \quad A_3' = \alpha^4 E_x A_3'' \tag{46}$$

where

$$A_1'' = \frac{2}{\lambda}(\gamma^4 + 6\gamma^2\rho^2 + \rho^4) + \frac{2\mu_{xy}}{E_x}(\gamma^2 + \rho^2)$$

$$A_2'' = 2\left(\frac{\sigma_{yx}}{\lambda} + \frac{\mu_{xy}}{E_x}\right)(\gamma^2 + \rho^2) \tag{47}$$

$$A_3'' = 2\frac{E_y}{E_x\lambda} + 2\frac{\mu_{xy}}{E_x}(\gamma^2 + \rho^2)$$

Using equations (32), (33), (42), (43), and (45) the following relations are found:

$$\frac{\phi}{A_4'} = \frac{\pi^2\lambda S_x}{J\rho^2\alpha^4 E_x A_4''}$$

$$\frac{\phi}{A_5'} = \frac{\pi^2\lambda S_y}{J\rho^2\alpha^4 E_x} \tag{48}$$

where

$$A_4'' = \gamma^2 + \rho^2 \tag{49}$$

Using equations (41) and (44) it is found that

$$r = \frac{\pi^2}{hJ\rho^2\alpha^2} \tag{50}$$

Substitution of the expressions given by equations (46), (48), and (50) into the second part of equation (40) and dividing by $1 - \frac{c}{h}$ yields:

$$\psi_2 = \frac{\pi^2}{(1 - \frac{c}{h})\, 4J\rho^2\gamma} \left\{ \frac{\frac{I}{h^3}\left[A_1'' + 2A_2'' + A_3'' + (A_1''A_3'' - A_2''^2)\frac{\pi^2\lambda}{J\rho^2}\left(\frac{S_x}{A_4''} + S_y\right)\right]}{1 + \frac{\pi^2\lambda}{J\rho^2}\left(S_x\frac{A_1''}{A_4''} + S_y A_3''\right) + \frac{\pi^4\lambda^2}{J^2\rho^4} S_x S_y \frac{A_1''A_3'' - A_2''^2}{A_4''}} + \frac{I_f}{h^3}(A_1'' + 2A_2'' + A_3'') \right\} \tag{51}$$

Thus equation (40) can be written

$$K_f = \psi_1 + \psi_2 \tag{52}$$

which is substituted in equation (39) to obtain the shear stress in the facings (τ_f) at the critical load. $\underline{\psi_1}$ is a function of $\underline{J}$, $\underline{\gamma}$, $\underline{\rho}$, and the elastic properties of the facings and $\underline{\psi_2}$ is a function of $\underline{\frac{c}{h}}$, $\underline{J}$, $\underline{\frac{I}{h^3}}$, $\underline{\frac{I_f}{h^3}}$, $\underline{S_x}$, $\underline{S_y}$, $\underline{\gamma}$, $\underline{\rho}$, and the elastic properties of the facings. The values of $\underline{\gamma}$ and $\underline{\rho}$ are chosen so as to obtain minimum values of $\underline{K_f}$.

If the facings are of equal thickness

$$\frac{I}{h^3} = \frac{1}{16}\left(1 - \frac{c}{h}\right)\left(1 + \frac{c}{h}\right)^2 \tag{53}$$

$$\frac{I_f}{h^3} = \frac{1}{48}\left(1 - \frac{c}{h}\right)^3 \tag{54}$$

and the number of parameters is reduced by two.

The values given by equations (53) and (54) are not greatly in error when the facings are not of equal thickness. They are only ten percent too great when one facing is twice the thickness of the other.

Isotropic Facings of Equal Thickness and Orthotropic Core

For isotropic facings of equal thickness some simplification can be made because of the following relations:

$$
\begin{aligned}
E_x &= E_y = E \\
\sigma_{xy} &= \sigma_{yx} = \sigma \\
\mu_{xy} &= \frac{E}{2\,(1+\sigma)}
\end{aligned}
\tag{55}
$$

Substituting these values in expression (44)

$$
\psi_1 = \frac{J\rho^2}{4\pi^2\gamma}\left[\frac{(\gamma+\rho)^4}{\left[(\gamma+\rho)^2+1\right]^2} + \frac{(\gamma-\rho)^4}{\left[(\gamma-\rho)^2+1\right]^2}\right] \tag{56}
$$

Substitution of the values given by equations (55) into equations (47) and taking $\underline{\sigma}$ to be 1/4 yields:

$$
\begin{aligned}
A_1'' &= \frac{2}{\lambda}\left(\gamma^4 + 6\gamma^2\rho^2 + \rho^4\right) + \frac{3}{4\lambda}\left(\gamma^2+\rho^2\right) \\
A_2'' &= \frac{5}{4\lambda}\left(\gamma^2+\rho^2\right) \\
A_3'' &= \frac{2}{\lambda} + \frac{3}{4\lambda}\left(\gamma^2+\rho^2\right)
\end{aligned}
\tag{57}
$$

Substitution of the values given by equations (49), (53), (54), and (57) into equation (51) yields

$$\psi_2 = \frac{\pi^2 (1 + \frac{c}{h})^2}{32\lambda J\rho^2\gamma} \; \frac{(1+\gamma^2+\rho^2)^2 + 4\gamma^2\rho^2 + \frac{\pi^2 S_x}{J\rho^2}\left[1+\theta(\gamma^2+\rho^2)\right]\left\{\frac{3}{8}\left[(1+\gamma^2+\rho^2)^2+4\gamma^2\rho^2\right] + \frac{4\gamma^2\rho^2}{\gamma^2+\rho^2}\right\}}{1+\frac{\pi^2 S_x}{J\rho^2}\left\{\gamma^2+\rho^2+\theta+\frac{4\gamma^2\rho^2}{\gamma^2+\rho^2}+\frac{3}{8}\left[1+\theta(\gamma^2+\rho^2)\right]\right\}+\frac{\pi^4 S_x^2}{J^2\rho^4}\theta\left\{\frac{3}{8}\left[(1+\gamma^2+\rho^2)^2+4\gamma^2\rho^2\right]+\frac{4\gamma^2\rho^2}{\gamma^2+\rho^2}\right\}}$$

$$+ \frac{\pi^2 (1 - \frac{c}{h})^2}{96\lambda J\rho^2\gamma}\left[(1+\gamma^2+\rho^2)^2 + 4\gamma^2\rho^2\right] \qquad (58)$$

where

$$\theta = \frac{S_y}{S_x} = \frac{\mu'_{\zeta x}}{\mu'_{y\zeta}}$$

The value of the critical shear stress in the facings is given by:

$$\tau_f = K_f \, E \, \frac{h}{r} \qquad (59)$$

where

$$K_f = \psi_1 + \psi_2$$

and values of $\underline{\psi_1}$ and $\underline{\psi_2}$ are taken from equations (56) and (58). The values of $\underline{\gamma}$ and $\underline{\rho}$ are chosen to obtain the least value of $\underline{K_f}$.

Maximum Buckling Stress Associated with Very Thin Facings

If the facings are very thin, their individual flexural stiffnesses may be neglected. The critical shear stress will then be limited by that obtained when the cylinder wall buckles into a great number of very short waves. This critical stress is associated with the instability of the core in shear. Equations (41) and (42) show that the value of expression

$$J\rho^2 = \frac{\pi^2 r}{n^2 h}$$

becomes very small when the number of buckles n becomes very large and, therefore, the value of ψ_1 may be neglected. Also only the expressions in the first term of ψ_2 that have the expression $J\rho^2$ in their denominators need be considered. Further, because the facings are very thin the expression $(1 - \frac{c}{h})^2$ in the second term of ψ_2 is substantially zero, so this term may be neglected and the expression $(1 + \frac{c}{h})^2$ in the first term of ψ_2 becomes 4. Equation (59) may then be written

$$\tau_f = \frac{Eh}{8\lambda r S_x} \frac{1 + \theta(\gamma^2 + \rho^2)}{\theta\gamma} \tag{60}$$

Minimum values of τ_f occur when $\rho = 0$ and $\gamma = \frac{1}{\sqrt{\theta}}$. Substituting these values and the expression for θ in (60):

$$\tau_f = \frac{Eh}{4\lambda r\sqrt{S_x S_y}} \tag{61}$$

Substitution of the expressions for S_x and S_y given by equation (45) and remembering that $E_x = E$ and that c is substantially equal to h

$$\tau_f = \frac{h}{2f}\sqrt{\mu'_{\zeta x}\,\mu'_{y\zeta}} \tag{62}$$

This is the maximum critical shear stress that can be obtained if the individual stiffnesses of the facings are neglected.

Very Long Cylinders

When the cylinder is very long, it is convenient to eliminate the parameter $\underline{J}$ by means of the relation:

$$\frac{J\rho^2}{\pi^2} = \frac{\rho}{n^2 h} \tag{63}$$

obtained from equations (41) and (42). As the cylinder is lengthened, the number of buckles $\underline{n}$ becomes two (10) and thus according to equation (41) the value of $\underline{\rho}$ approaches zero. Thus equation (44) becomes:

$$\psi_1 = \frac{r}{8h} \frac{\gamma^3}{\frac{E_x}{E_y}\gamma^4 + \left(\frac{E_x}{\mu_{xy}} - 2\sigma_{xy}\right)\gamma^2 + 1} \tag{64}$$

Equations (47) and (49) for substitution in equation (51) become:

$$A_1'' = \frac{2}{\lambda}\gamma^4 + \frac{2\mu_{xy}}{E_x}\gamma^2$$

$$A_2'' = 2\left(\frac{\sigma_{yx}}{\lambda} + \frac{\mu_{xy}}{E_x}\right)\gamma^2 \tag{65}$$

$$A_3'' = 2\frac{E_y}{E_x\lambda} + 2\frac{\mu_{xy}}{E_x}\gamma^2$$

$$A_4'' = \gamma^2$$

Minimum values of $\underline{K_f}$ from equation (52), using these new expressions for $\underline{\psi_1}$ and $\underline{\psi_2}$, are found by choosing suitable values of $\underline{\gamma}$.

If the facings are isotropic, equation (44) becomes:

$$\psi_1 = \frac{r}{8h} \frac{\gamma^3}{(\gamma^2 + 1)^2} \tag{66}$$

and equation (51) becomes:

$$\psi_2 = \frac{2h}{\lambda\,(1 - \frac{c}{h})\,r}\;\frac{(1+\gamma^2)^2}{\gamma}\left[\frac{I}{h^3}\;\frac{1 + \frac{3n^2hS_x}{8r}(1+\theta\gamma^2)}{1 + \frac{n^2hS_x}{r}\left[\gamma^2 + \theta + \frac{3}{8}(1+\theta\gamma^2)\right] + \frac{3n^4h^2\theta}{8r^2}(1+\gamma^2)^2} + \frac{I_f}{h^3}\right] \tag{67}$$

If the core is also isotropic, equation (66) is not changed but equation (67) becomes:

$$\psi_2 = \frac{2h}{\lambda(1 - \frac{c}{h})r}\;\frac{(1+\gamma^2)^2}{\gamma}\left[\frac{\frac{I}{h^3}}{1 + \frac{n^2hS}{r}(1+\gamma^2)} + \frac{I_f}{h^3}\right] \tag{68}$$

Values of γ are chosen that make $K_f = \psi_1 + \psi_2$ a minimum.

The shear force per unit length of the edge of the sandwich cylinder is given by:

$$N = \tau h = \tau_f\,(1 - \frac{c}{n})h = (1 - \frac{c}{n})K_f E\,\frac{h^2}{r} = (1 - \frac{c}{n})(\psi_1 + \psi_2)E\,\frac{h^2}{r} \tag{69}$$

The expression for N obtained by substituting the expressions for ψ_1 and ψ_2 given by equations (66) and (68), omitting the last term in the brackets in equation (68), can be shown to be in agreement with the expression obtained by Gerard (3) by another method. The omitted term is associated with the flexural rigidities of the facings and may add considerably to the critical load if the core has a small modulus of rigidity and the facings are thick.

Numerical Results

Numerical values of the buckling coefficient K_f for cylinders having isotropic facings of equal thickness were obtained by use of equation (59) and equations (56) and (58) and are plotted in figures 3 to 8. These curves are reasonably accurate for sandwich constructions having unequal facings if the thickness of one facing is not more than twice that of the other. Values are plotted for J ranging from 10 to 10,000, S_x from 0 to 2, for θ equal to 0.4, 1.0, and 2.5

and $\frac{c}{h}$ equal to 1 and 0.7. The particular values of θ were chosen to agree with the properties of honeycomb cores of hexagonal cells and isotropic cores. Estimates of K_f for other values of θ can be obtained by interpolation. Values of K_f for $\frac{c}{h}$ equal to 1 apply to sandwich panels having facings so thin that their bending stiffnesses may be neglected. An estimate of the effect of facing thickness is given by the values of K_f for $\frac{c}{h}$ equal to 0.7.

In calculating the minimum values of the buckling coefficient (K_f), it was assumed that the values of ρ were continuous. Strictly speaking, the values of ρ are discrete because they contain an integral number of buckles n in the circumference of the cylinder as shown by equation (41). These discrete values depend upon the ratio of the radius of the cylinder to its length.

The results of the calculation showed that the values of the buckling coefficient (K_f) were not sensitive to the values of ρ; thus, the error introduced by the assumption that ρ is continuous is small.

As the cylinder becomes longer the number of buckles becomes less and is equal to two for an infinite length (10). Thus, for large values of J the number of buckles approaches two. The results of the calculations showed that for J equal to 10,000, values of ρ are about 0.1 for substantially all of the calculations made (ρ was less than 0.1 for θ equal to 2.5 when S_x was large). Substitution of these values in equation (63) shows that for two waves the ratio of the radius of the cylinder to its thickness is about 40 and the assumption of a greater number of waves leads to a larger value. Thus, for large values of J these curves apply only to cylinders with a ratio of radius to thickness in the neighborhood of 40 or larger. If this ratio is less than 40, the curves give conservative values of the buckling coefficient.

Numerical values of K_f for infinitely long isotropic cylinders having equal facings were calculated from the sum of equations (66) and (68) using equations (53) and (54) and are plotted in figure 9 against S for different values of $\frac{r}{h}$. The solid lines apply to a $\frac{c}{h}$ value of unity. The uppermost solid line that cuts across the others indicates shear instability of the core and is obtained from equation (62). The dotted lines apply to a $\frac{c}{h}$ value of 0.7.

Conclusions

The shear stress in the facings at which buckling[6] of a cylinder of sandwich construction in torsion will occur is given by:

$$\tau_f = K_f E_f \frac{h}{r}$$

For orthotropic facings and cores, $\underline{K_f}$ is given by equation (52) in connection with equations (44) and (51). For isotropic facings and orthotropic cores, equations (56) and (58) are used in place of (44) and (51). Numerical values have been calculated and are plotted in figures 3 to 8. Maximum values for the critical shear stress for cylinders having orthotropic cores and very thin isotropic facings are given by equation (62), and are plotted as the upper solid curve in figure 9.

Values of $\underline{K_f}$ for very long sandwich cylinders having orthotropic facings and cores is given by equation (52), using equation (64) and equations (65) in equation (51) for values of $\underline{\psi_1}$ and $\underline{\psi_2}$. Values for very long cylinders having isotropic facings and orthotropic cores are given by the sum of equations (66) and (67). Values for isotropic cylinders are given by the sum of equations (66) and (68). Numerical values of this sum are plotted in figure 9. These values are limited by equation (62) as shown in the figure.

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[6] The possibility of wrinkling at a lower stress (similar to that described in Forest Products Laboratory Report No. 1810), (9) should be investigated.

Notation

a_1, a_2	defined by equation (16).
A	$\frac{1}{E_b}$.
A_1', A_2', A_3', A_4', A_5'	defined by equations (29) to (33).
A_1'', A_2'', A_3'', A_4''	defined by equations (47) and (49).
b	length of cylinder.
B	$\frac{1}{E_a}$.
c	thickness of core.
C	$\frac{1}{\mu_m} - \frac{2\sigma_{xy}}{E_a}$.
e_{xx}, e_{xy}, etc.	components of strain.
E_x, E_y, E	Young's moduli of the facings.
E_a	$E_x (1 - \frac{c}{h})$.
E_b	$E_y (1 - \frac{c}{h})$.
f_1, f_2	thickness of the facings.
g	coefficient in equation (1).
h	$c + f_1 + f_2$.
H	defined by equations (28) and (34).
I	defined by equation (25).
I_f	defined by equation (26).
J	$\frac{b^2}{rh}$.

k, k'	parameters introduced by equations (23).
K, K_f	buckling coefficients (see equations (39).
K_1, K_2	defined by equations (17) and (18).
L	$E_x\sigma_{yx} + 2\lambda\mu_{xy}$.
n	number of buckles in the circumferential direction.
q, q'	parameters introduced in equations (23).
r	radius of middle surface of the cylindrical shell.
S_x, S_y	defined by equations (45).
u	axial component of displacement.
v	circumferential component of displacement.
w	radial component of displacement.
W_1	extensional strain energy.
W_2	flexural strain energy.
W_3	work done by applied couple.
X_x, X_y, etc.	components of stress.
α	$\frac{n}{r}$.
β	$\frac{\pi}{b}$.
ϕ	defined by equation (27).
γ	inclination of buckles.
δ	defined by equation (4).
ϵ	defined by equation (4).
ζ	coordinate shown in figure 2.

λ	$1 - \sigma_{xy}\sigma_{yx}$.
μ_{xy}	modulus of rigidity of facings.
$\mu'_{\zeta x}$, $\mu'_{y\zeta}$	moduli of rigidity of core.
μ_m	$\mu_{xy}\,(1 - \frac{c}{h})$.
σ_{xy}, σ_{yx}	Poisson's ratios of the facings.
ρ	$\frac{\beta}{\alpha} = \frac{\pi r}{nb}$
τ	mean shear stress.
τ_f	shear stress in facings.

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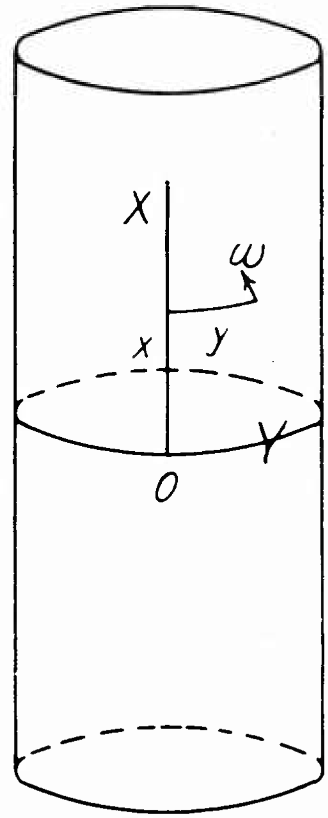

Figure 1.--Choice of coordinates on the surface of a cylinder.

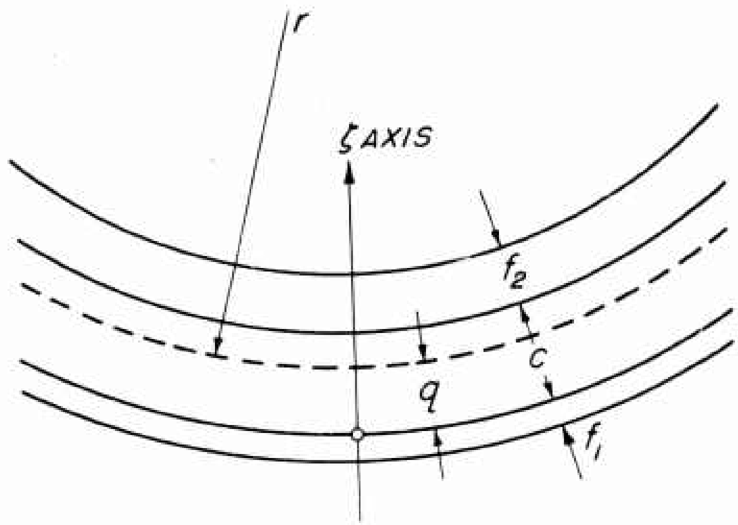

Figure 2. --Section of a cylindrical shell, where r is the radius of the middle surface of the shell, c is the thickness of the core, f_1 and f_2 are the thicknesses of the facings, q is the distance indicated in the figure, and ζ is the coordinate indicated in the figure.
Note: The curved lines are arcs of concentric circles.

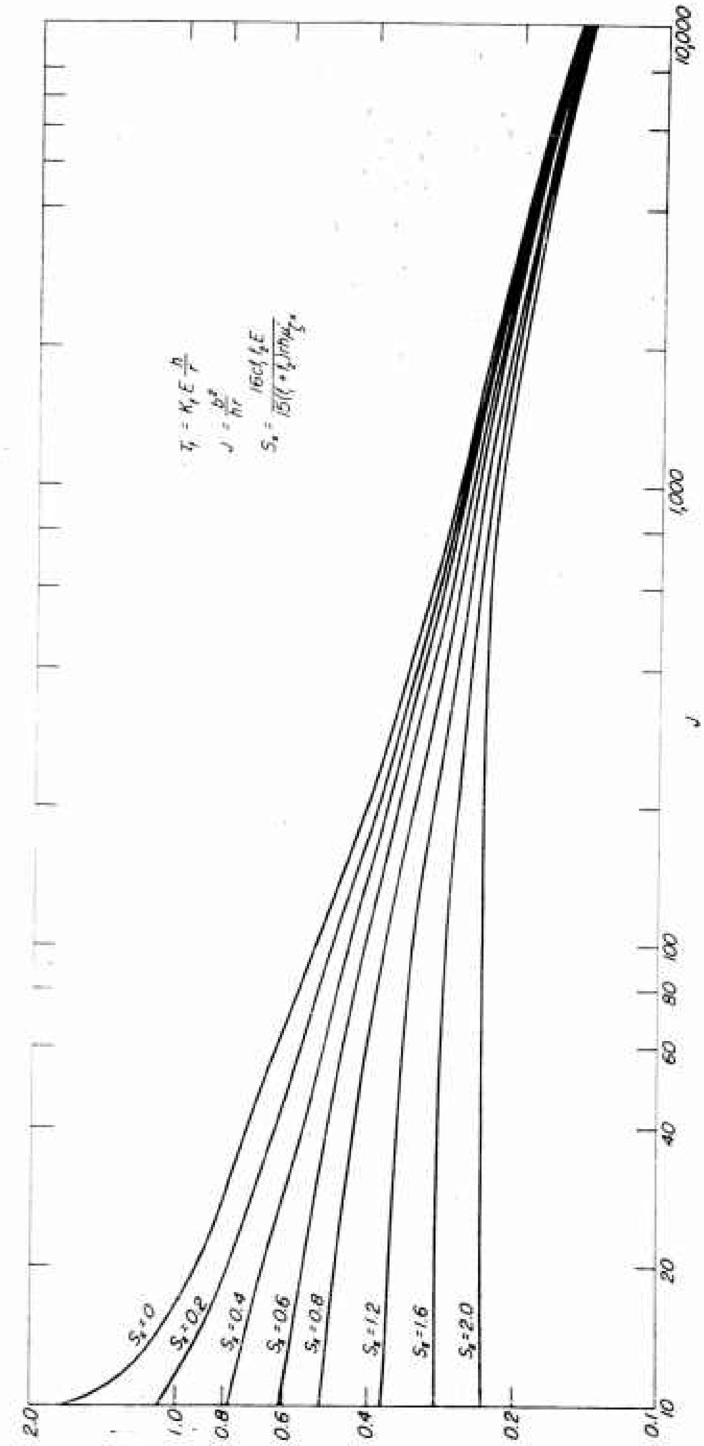

Figure 3. --Buckling coefficients for cylinders having isotropic facings $\frac{c}{h} = 1$, $\theta = \frac{\mu'_{\xi x}}{\mu_{y\xi}} = 0.4$.

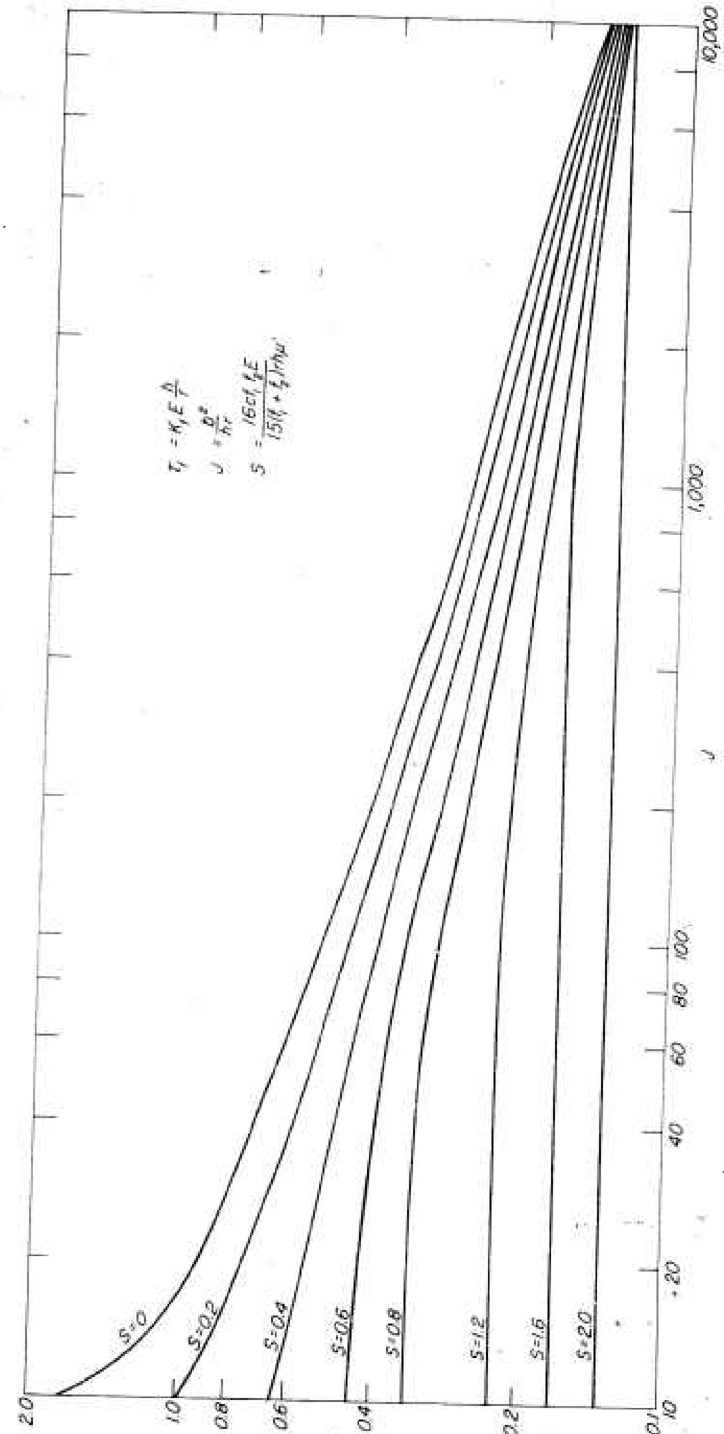

Figure 4.[illegible]-Buckling coefficients for isotropic cylinders $\frac{c}{h} = 1$, $\theta = \frac{\mu'_{\xi x}}{\mu'_{y\xi}} = 1$.

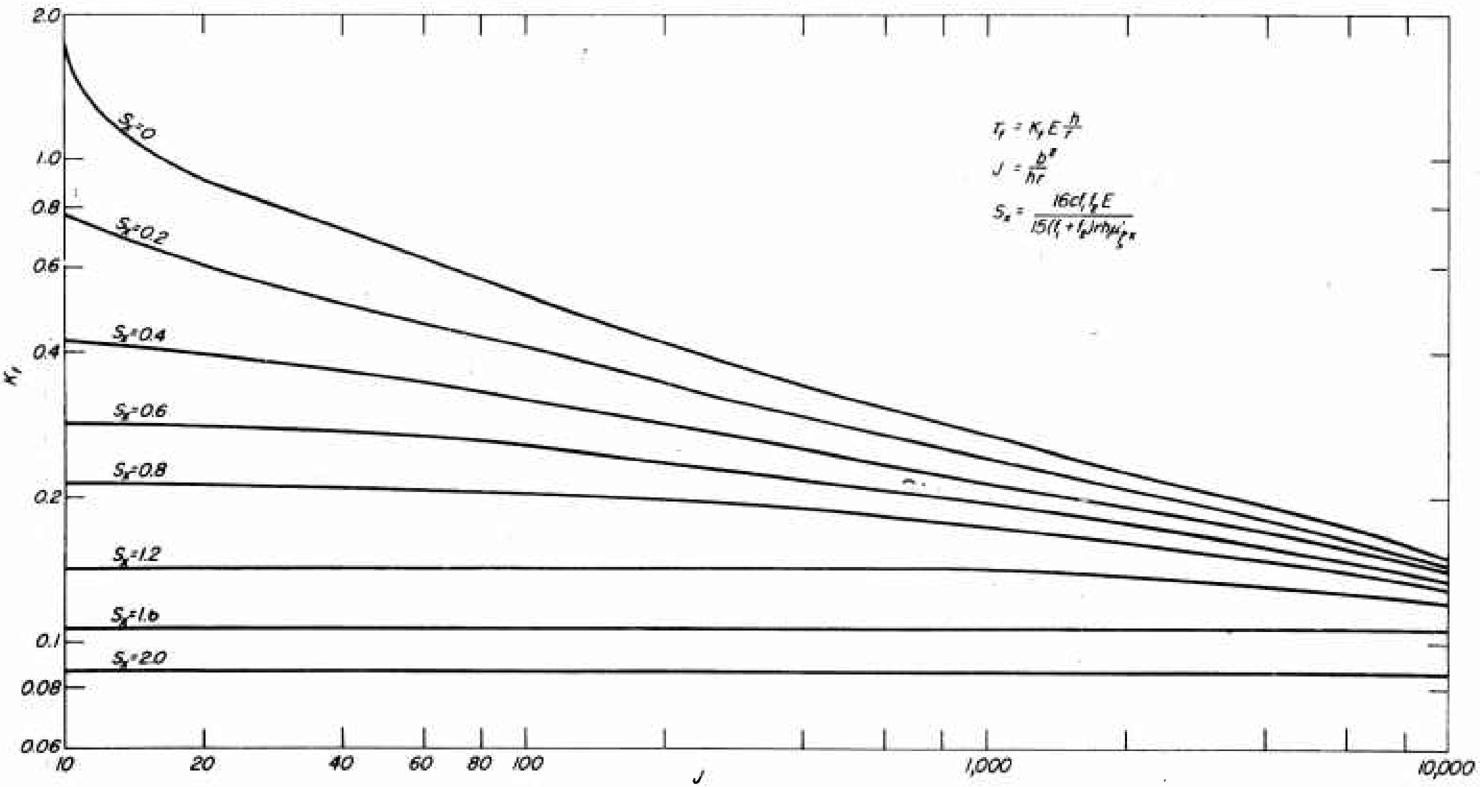

Figure 5. --Buckling coefficients of cylinders having isotropic facings $\frac{c}{h} = 1$, $\theta = \frac{\mu'_{\xi x}}{\mu'_{y\xi}} = 2.5$.

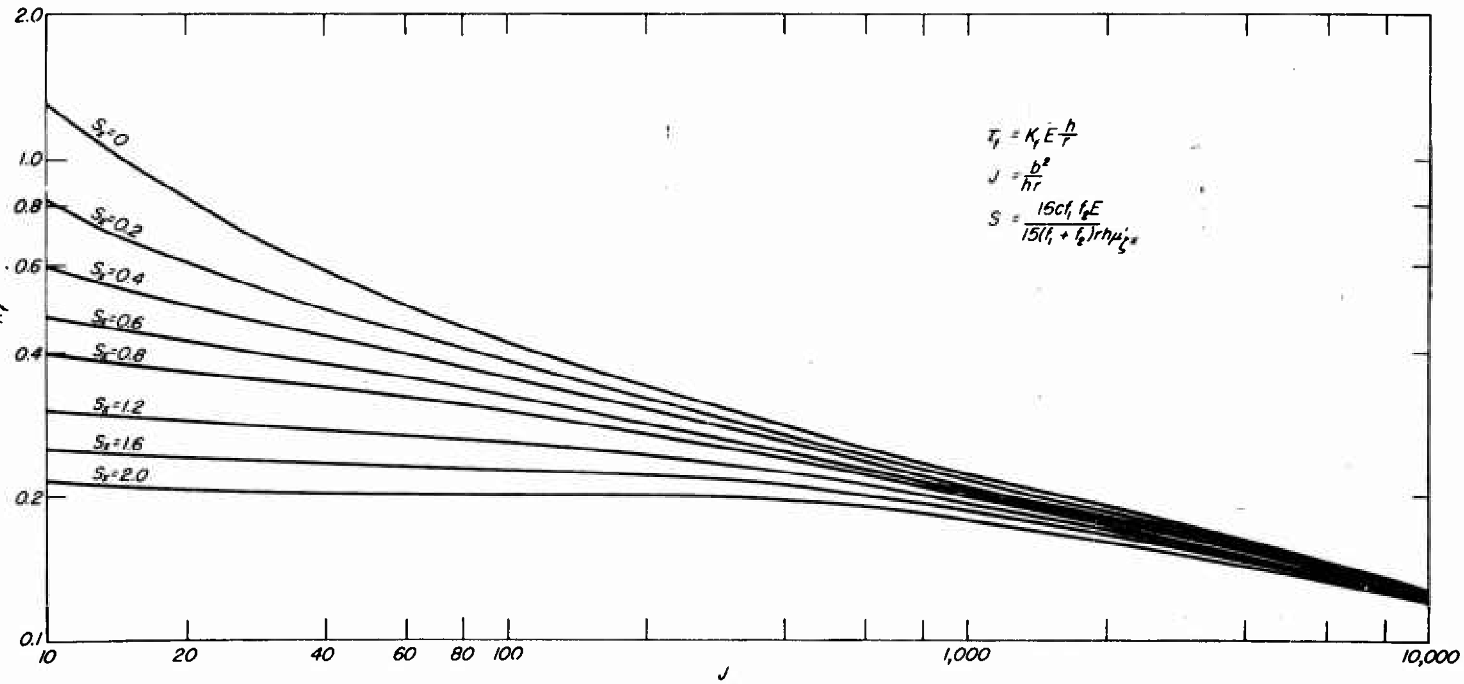

Figure 6. --Buckling coefficients of cylinders having isotropic facings $\frac{c}{h} = 0.7$, $\theta = \frac{\mu'_{\zeta x}}{\mu'_{y\zeta}} = 0.4$.

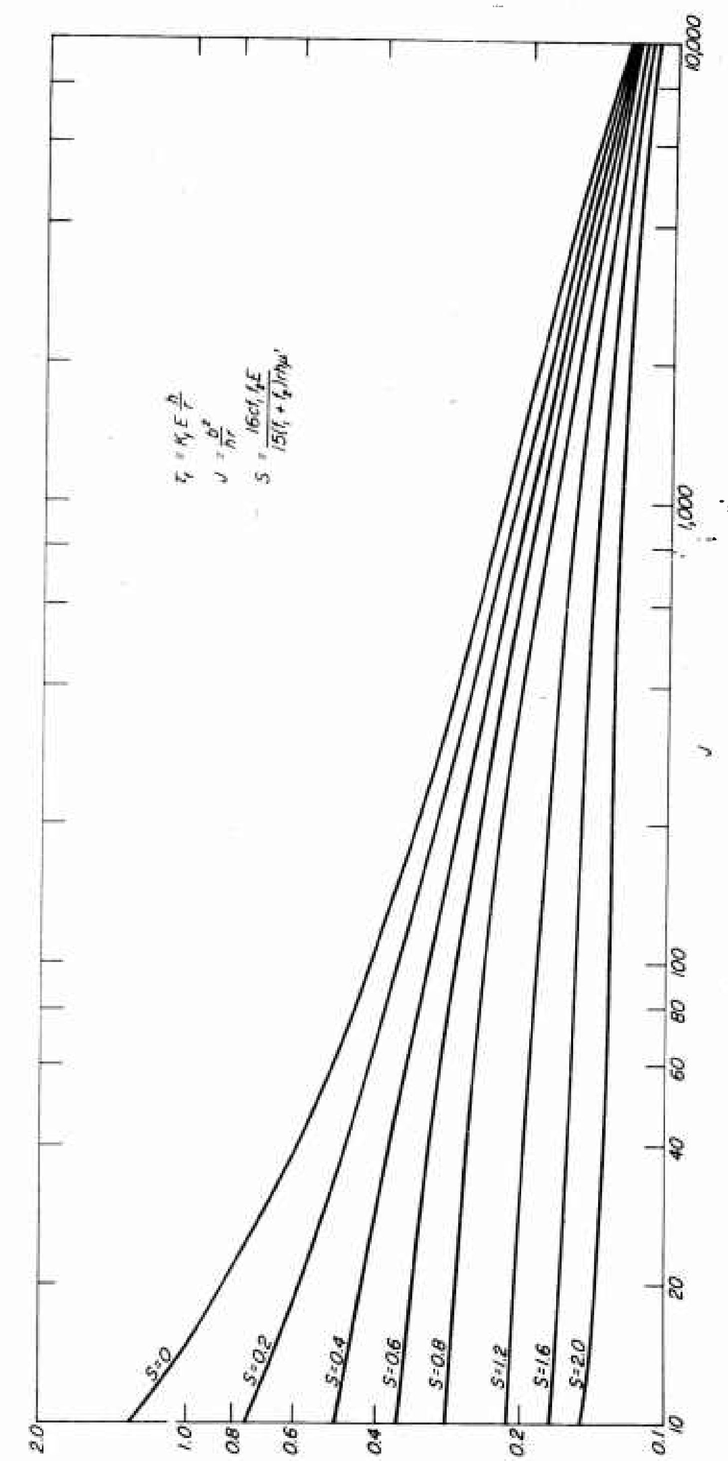

Figure 7. --Buckling coefficients for isotropic cylinders $\frac{c}{h} = 0.7$, $\theta = \frac{\mu'_{\xi x}}{\mu'_{y\xi}} = 1$.

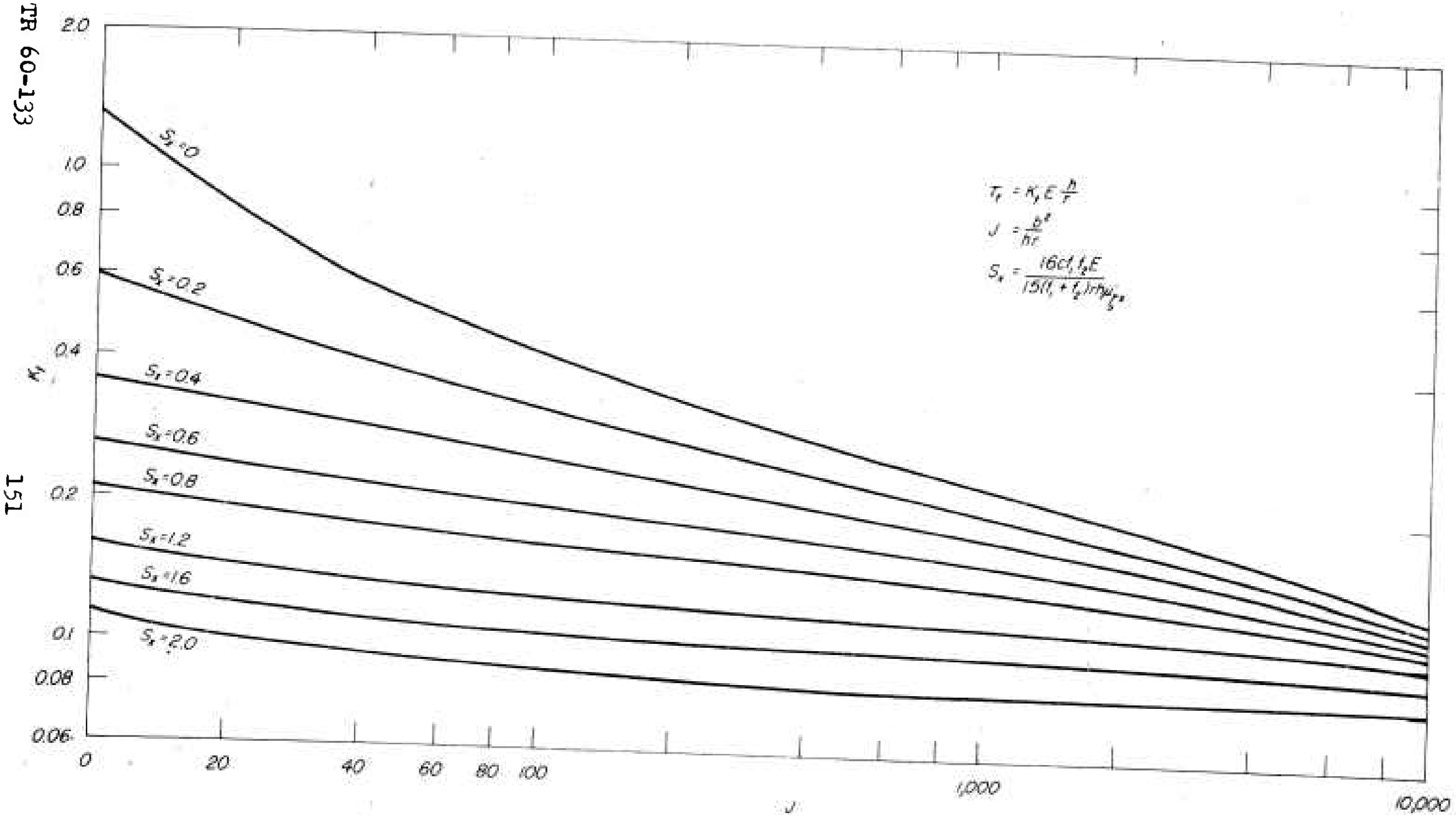

Figure 8. --Buckling coefficients of cylinders having isotropic facings $\frac{c}{h} = 0.7$, $\theta = \frac{\mu'_{\xi x}}{\mu'_{y\xi}} = 2.5$.

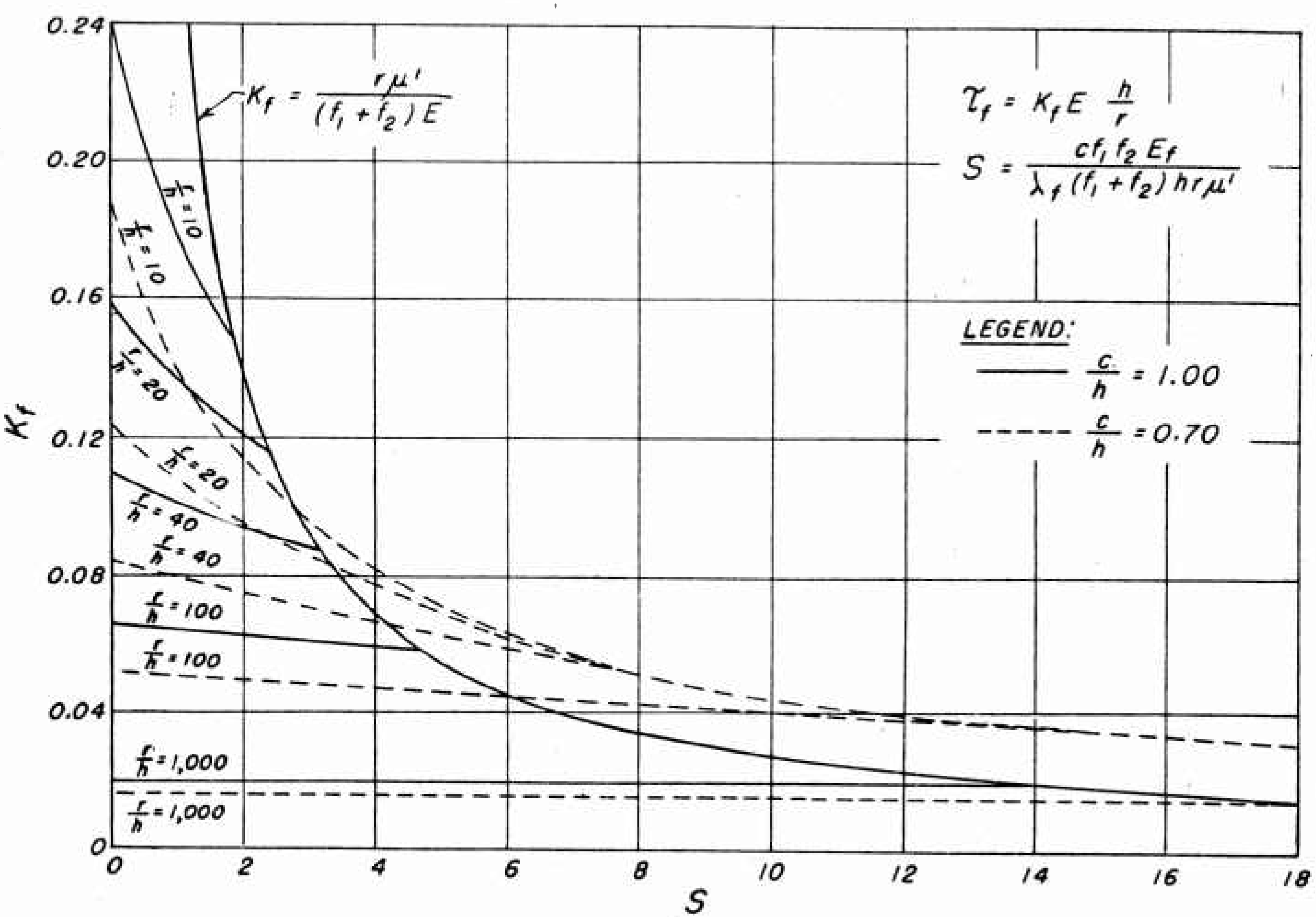

Figure 9. --Buckling coefficients for infinitely long cylinders of isotropic sandwich constructions in torsion.

SECTION V

ANALYSIS OF LONG CYLINDERS OF SANDWICH CONSTRUCTION UNDER UNIFORM EXTERNAL LATERAL PRESSURE[1]

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Summary

A theoretical solution is obtained for the stresses in long sandwich cylinders subjected to externally applied uniform lateral pressure. The analysis is extended to take into account failure due to elastic instability, and an equation is derived for the determination of critical loads on long sandwich cylinders. The application of this analysis to long curved panels (portions of a long cylinder) is discussed. The sandwich construction is assumed to consist of isotropic membrane facings and an orthotropic core.

Introduction

In this report it is assumed that the sandwich cylinder is comprised of thin, isotropic facings of a relatively stiff material separated by an orthotropic core of a relatively weak material. The facings are assumed to be so thin that bending and shear in the individual facings may be neglected. It is further assumed that the only stress components present in the core are the normal stress on surfaces parallel to the facings and the transverse shear stress. Since elasticity theory is used in regard to the core, the solution is not limited with respect to core thickness. Results based on the assumption of membrane facings are somewhat limited

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[1] This report is one of a series prepared by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics Order No. 01595.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

in applicability, but they should be sufficiently accurate for the majority of sandwich panels. It is felt that the core assumptions represent reasonably well the cores presently in use, especially those of the honeycomb type.

The method of analysis is based primarily on the fact that the core assumptions permit the direct determination of the core displacement functions insofar as their dependence on the radial distance is concerned. The usual requirement of continuity of displacements at the interfaces is specified. The stability analysis is based on the method used by Timoshenko[3] in the analysis of the stability of homogeneous isotropic cylinders. The stresses in the sandwich cylinder are determined for loads less than the critical load; and, in discussing buckling, only small deformations from this uniformly compressed form of equilibrium are considered.

Notation

r, θ, z	radial, tangential, and longitudinal coordinates, respectively
a	radius to middle surface of outer facing
b	radius to middle surface of inner facing
t	thickness of facings
E	modulus of elasticity of facings
μ	Poisson's ratio of facings
E_c	modulus of elasticity of core in direction normal to facings
$G_{r\theta}$	modulus of rigidity of core in r-θ plane
q	intensity of uniform external lateral loading
p	intensity of uniform external lateral pressure, equal to $-q$
σ_r, σ'_r, σ_{rc}	normal stress in the radial direction in outer facing, inner facing, and core respectively
$\bar{\sigma}_r$, $\bar{\sigma}'_r$, $\bar{\sigma}_{rc}$	small normal stresses in the radial direction in outer facing, inner facing, and core respectively
$\bar{\tau}_{r\theta}$, $\bar{\tau}'_{r\theta}$	small shear stress in the plane of the middle surface of the outer and inner facing, respectively

[3] S. Timoshenko. Theory of Elastic Stability, p. 446.

$\bar{\tau}_{r\theta c}$ small transverse shear stress in the core

N_θ, N_θ' direct stress resultants in the tangential direction in the plane of outer and inner facing, respectively

$\bar{N}_\theta$, $\bar{N}_\theta'$ small direct stress resultants in the tangential direction in the plane of outer and inner facing, respectively

u, u', u_c radial displacements of outer facing, inner facing, and core, respectively

$\bar{u}$, $\bar{u}'$, $\bar{u}_c$ small radial displacements of outer facing, inner facing, and core, respectively

$\bar{v}$, $\bar{v}'$, $\bar{v}_c$ small tangential displacements of outer facing, inner facing, and core, respectively

ϵ_θ, ϵ_θ' tangential strains in outer and inner facing, respectively

$\bar{\epsilon}_\theta$, $\bar{\epsilon}_\theta'$, $\bar{\epsilon}_{\theta c}$ small tangential strains in outer facing, inner facing, and core, respectively

n number of waves in circumference of cylinder

α one-half the central angle of curved panel

k $$\frac{1}{(1 + b/a) - \dfrac{Et \log b/a}{E_c a}}$$

log natural logarithm

A, B, A_n', B_n', C_n', H_n', A_n, B_n, C_n, H_n arbitrary constants

Theoretical Analysis

The first step in the analysis is that of determining the stresses in the sandwich cylinder for pressures up to the critical pressure. The cylinder remains circular and is in a state of uniform compression. Throughout the analysis, q represents a uniform, lateral load acting in the positive radial direction. The pressure, p, is equal to -q.

Equilibrium of Core

As previously stated, it is assumed that the core can transmit only normal stresses in the radial direction and transverse shear stresses. When the cylinder is in a state of uniform compression, the only stress present in the core is the normal stress in the radial direction, σ_{rc}. Considering the equilibrium of the differential element of the core shown in figure 1, the summation of forces in the radial direction results in the following equation:

$$- \sigma_{rc} \; r \; d\theta \; dz + (\sigma_{rc} + \frac{d\sigma_{rc}}{dr} dr)(r + dr) \; d\theta \; dz = 0$$

This reduces to the following differential equation of equilibrium:

$$\frac{d\sigma_{rc}}{dr} + \frac{\sigma_{rc}}{r} = 0 \qquad (1)$$

The solution of equation (1) is

$$\sigma_{rc} = E_c \frac{A}{r} \qquad (2)$$

in which $E_c A$ represents the constant of integration. The radial displacement of the core, u_c, is related to the radial stress, σ_{rc}, by the following equation:

$$\sigma_{rc} = E_c \frac{du_c}{dr} \qquad (3)$$

From equation (2) it follows that

$$\frac{du_c}{dr} = \frac{A}{r}$$

Integration gives

$$u_c = A \log r/a + B \qquad (4)$$

The use of log r/a instead of log r merely alters the arbitrary constant of integration.

Equilibrium of Facings

With reference to the differential elements of the facings shown in figure 2, it is seen that, when the cylinder is uniformly stressed by the action of the external load, q, the only stresses present in the facings are the radial stresses, σ_r and σ_r', which are exerted by the core, and the tensile forces per unit length of facing, N_θ and N_θ'. These stresses are assumed to be acting on the middle surfaces of the facings, an assumption which is justified since the facings are assumed to be thin. An equilibrium equation can be obtained for each facing by summing forces in the radial direction on each element. The equilibrium equation which pertains to the outer facing is

$$qa\,d\theta\,dz - \sigma_r a\,d\theta\,dz - N_\theta\,d\theta\,dz = 0$$

This reduces to

$$N_\theta = qa - \sigma_r a$$

or, since $\sigma_r = (\sigma_{rc})_{r = a}$

$$N_\theta = qa - a(\sigma_{rc})_{r = a} \tag{5}$$

In a similar manner, the equilibrium equation of the inner facing is obtained as

$$N_\theta' = b(\sigma_{rc})_{r = b} \tag{6}$$

The requirement is now made that the displacements of the core and facings be equal at the respective interfaces. It is assumed that the core extends to the middle surfaces of the facings. Thus

$$u = (u_c)_{r = a} \tag{7}$$

and

$$u' = (u_c)_{r = b} \tag{8}$$

From Hooke's law,

$$N_\theta = Et(\epsilon_\theta)$$

Since $\epsilon_\theta = \frac{u}{a}$, and, in view of equation (7), $\epsilon_\theta = \frac{(u_c)_{r = a}}{a}$, then

$$N_\theta = Et\frac{1}{a}\,(u_c)_{r = a}$$

By using equation (4) in conjunction with the above equation

$$N_\theta = Et\left(\frac{B}{a}\right) \tag{9}$$

Also, from Hooke's law,

$$N_\theta' = Et(\epsilon_\theta')$$

Since $\epsilon_\theta' = \frac{u'}{b}$, the above equation in conjunction with equations (8) and (4) leads to:

$$N_\theta' = Et\left(\frac{A}{b} \log b/a + \frac{B}{b}\right) \tag{10}$$

Since, from equation (2), $(\sigma_{rc})_{r = a} = E_c\frac{A}{a}$ and $(\sigma_{rc})_{r = b} = E_c\frac{A}{b}$, equations (5) and (6) may be written as

$$N_\theta = qa - E_c A \tag{11}$$

and

$$N_\theta' = E_c A \tag{12}$$

After equating the right-hand sides of equations (9) and (11) and the right-hand sides of equations (10) and (12), two equations are obtained from which the values of the constants A and B may be determined. These two equations are

$$qa - E_c A = \frac{Et}{a}B \tag{13}$$

and

$$E_c A = \frac{Et}{b}(A \log b/a + B) \tag{14}$$

From equation (14)

$$B = \left[\frac{E_c b}{Et} - \log b/a\right] A \tag{15}$$

Substituting the above value of B into equation (13) and multiplying through by $\frac{a}{Et}$,

$$\frac{qa^2}{Et} - \frac{E_c a}{Et} A = \left[\frac{E_c b}{Et} - \log b/a\right] A$$

from which

$$A = \frac{\frac{qa^2}{Et}}{\frac{E_c a}{Et}(1 + b/a) - \log b/a}$$

or

$$A = \frac{qa}{E_c}\left[\frac{1}{(1 + b/a) - \frac{Et \log b/a}{E_c a}}\right] \tag{16}$$

Substitution of the above value of A into equation (15) gives

$$B = \frac{qab}{Et}\left[\frac{1 - \frac{Et \log b/a}{E_c b}}{(1 + b/a) - \frac{Et \log b/a}{E_c a}}\right] \tag{17}$$

When the value of A given by equation (16) is substituted into equations (2), (11), and (12), respectively, the following expressions for the stresses in the cylinder are obtained:

$$\sigma_{rc} = \frac{qa}{r}(k) \tag{18}$$

$$N_\theta = qa\ (1 - k) \tag{19}$$

$$N_\theta' = qak \tag{20}$$

where $k = \dfrac{1}{(1 + b/a) - \dfrac{Et \log b/a}{E_c a}}$

Also, by using the values of A and B given by equations (16) and (17), equation (4) for the core displacement, after some simplification, becomes

$$u_c = \frac{qa^2}{Et}\left[(1 - k) + \frac{Et}{E_c a} k \log r/a\right] \tag{21}$$

Since u, the radial displacement of the outer facing, equals $(u_c)_{r = a}$ and u', the radial displacement of the inner facing, equals $(u_c)_{r = b}$,

$$u = \frac{qa^2}{Et}(1 - k) \tag{22}$$

and

$$u' = \frac{qa^2}{Et}(k\ b/a) \tag{23}$$

The analysis thus far is not limited strictly to membrane facings. The only restrictions on facing thicknesses are those imposed by the usual thin-walled-cylinder theory plus the additional requirement that the radial displacement of the middle surface of the facings be equal to the radial displacement of the corresponding interfaces.

The next step in the theoretical analysis concerns the development of the stability criteria. In this analysis, which follows, the facings are assumed to be membrane-type cylindrical shells. As mentioned previously, it is assumed that the stress situation in the deformed cylinder differs only slightly from the stress situation which exists just before buckling. A bar is placed over the appropriate symbol to denote the small stresses, strains, and displacements which occur when the cylinder goes from the initially uniformly stressed circular form to the slightly deformed configuration. These quantities are dependent upon θ as well as r.

Equilibrium of the Core

Since the cylinder is now considered to be in a slightly deformed state, the core is also slightly deformed. Figure 3 shows that, in addition to the radial stress, $q\frac{a}{r}k$, which exists just before buckling, an additional small radial stress, $\bar{\sigma}_{rc}$, and a small transverse shear stress, $\bar{\tau}_{r\theta c}$, must be considered. These stresses result when the core takes on small deformations from the initially uniformly compressed, circular shape, and they are dependent upon θ as well as r. The differential element shown in figure 3 is considered to be in equilibrium under the stresses shown. Summation of forces in the radial direction yields the following equilibrium equation:

$$- \left(q\frac{a}{r}k + \bar{\sigma}_{rc}\right) r\, d\theta\, dz + \left(q\frac{a}{r}k + \bar{\sigma}_{rc} + \frac{d(q\frac{a}{r}k)}{dr}dr + \frac{\partial\bar{\sigma}_{rc}}{\partial r}dr\right)(r + dr)\, d\theta\, dz$$

$$- \bar{\tau}_{r\theta c}\, dr\, dz + \left(\bar{\tau}_{r\theta c} + \frac{\partial\bar{\tau}_{r\theta c}}{\partial\theta}d\theta\right) dr\, dz = 0$$

If terms containing the products of more than three differentials are neglected, the above equation reduces to

$$\bar{\sigma}_{rc} dr\, d\theta\, dz + \frac{\partial \bar{\sigma}_{rc}}{\partial r} r\, dr\, d\theta\, dz + \frac{\partial \bar{\tau}_{r\theta c}}{\partial \theta} dr\, d\theta\, dz = 0$$

or, dividing through by r dr dθ dz,

$$\frac{\partial \bar{\sigma}_{rc}}{\partial r} + \frac{\bar{\sigma}_{rc}}{r} + \frac{1}{r} \frac{\partial \bar{\tau}_{r\theta c}}{\partial \theta} = 0 \tag{24}$$

Summation of forces in the tangential direction yields the following equation of equilibrium:

$$- \bar{\tau}_{r\theta c} r\, d\theta\, dz + (\bar{\tau}_{r\theta c} + \frac{\partial \bar{\tau}_{r\theta c}}{\partial r} dr)(r + dr)\, d\theta\, dz + \bar{\tau}_{r\theta c} dr\, d\theta\, dz = 0$$

which reduces to

$$\frac{\partial \bar{\tau}_{r\theta c}}{\partial r} + \frac{2\bar{\tau}_{r\theta c}}{r} = 0 \tag{25}$$

The core displacements are related to the core stresses by the following equations:

$$\bar{\sigma}_{rc} = E_c \frac{\partial \bar{u}_c}{\partial r} \tag{26}$$

and

$$\bar{\tau}_{r\theta c} = G_{r\theta} \left[\frac{1}{r} \frac{\partial \bar{u}_c}{\partial \theta} + \frac{\partial \bar{v}_c}{\partial r} - \frac{\bar{v}_c}{r} \right] \tag{27}$$

where $\bar{u}_c$ and $\bar{v}_c$ are the small radial and tangential core displacements from the uniformly compressed form of equilibrium. Equations (24 - 27) are sufficient to enable the direct determination of the displacement functions $\bar{u}_c$ and $\bar{v}_c$ insofar as their dependence on r is concerned. Thus, it is assumed that $\bar{u}_c$ and $\bar{v}_c$ are expressible in the following form:

$$\bar{u}_c = f_1(r) \cos n\theta \tag{28}$$

$$\bar{v}_c = f_2(r) \sin n\theta \tag{29}$$

This assumes that during buckling the circumference of the cylinder is subdivided into n waves; and the lowest value of the critical pressure will be obtained for n = 2, as in the case of homogeneous isotropic cylinders. The functions, $f_1(r)$ and $f_2(r)$, are now determined.

From equations (26), (27), (28), and (29) it is seen that $\bar{\sigma}_{rc}$ and $\bar{\tau}_{r\theta c}$ may be expressed as follows:

$$\bar{\sigma}_{rc} = f_3(r) \cos n\theta \tag{30}$$

$$\bar{\tau}_{r\theta c} = f_4(r) \sin n\theta \tag{31}$$

The use of the expression for $\bar{\tau}_{r\theta c}$ given by equation (31) in equation (25) results in the following equation:

$$\frac{d\, f_1(r)}{dr} + \frac{2\, f_4(r)}{r} = 0$$

The solution of the above equation is

$$f_4(r) = \frac{A_n'}{r^2}, \text{ where } A_n' \text{ is an arbitrary constant.}$$

Substitution of the above expression into equation (31) yields

$$\bar{\tau}_{r\theta c} = \frac{A_n'}{r^2} \sin n\theta \tag{32}$$

When the expressions for $\bar{\sigma}_{rc}$ and $\bar{\tau}_{r\theta c}$ given by equations (30) and (32) are substituted in equation (24), the result is

$$\frac{d\, f_3(r)}{dr} + \frac{f_3(r)}{r} = -\, n\frac{A_n'}{r^3}$$

The solution of the above equation is

$$f_3(r) = \frac{nA_n'}{r^2} + \frac{B_n'}{r}$$

and, from equation (30),

$$\bar{\sigma}_{rc} = \left(\frac{nA_n'}{r^2} + \frac{B_n'}{r}\right) \cos n\theta \tag{33}$$

Referring to equation (26), the following equation may be written:

$$\frac{\partial \bar{u}_c}{\partial r} = \frac{1}{E_c} \left(\frac{nA_n'}{r^2} + \frac{B_n'}{r}\right) \cos n\theta$$

Since, from equation (28), $\frac{\partial \bar{u}_c}{\partial r} = \frac{d\, f_1(r)}{dr} \cos n\theta$,

$$\frac{d\, f_1(r)}{dr} = \frac{1}{E_c} \left(\frac{nA_n'}{r^2} + \frac{B_n'}{r}\right)$$

Integrating

$$f_1(r) = \frac{1}{E_c} \left(- \frac{nA_n'}{r} + B_n' \log r + C_n'\right)$$

Therefore

$$\bar{u}_c = \frac{1}{E_c} \left(- \frac{nA_n'}{r} + B_n' \log r + C_n'\right) \cos n\theta \tag{34}$$

Rewriting equation (27),

$$\frac{\partial \bar{v}_c}{\partial r} - \frac{\bar{v}_c}{r} = \frac{\bar{\tau}_{r\theta c}}{G_{r\theta}} - \frac{1}{r} \frac{\partial \bar{u}_c}{\partial \theta}$$

Substituting for $\bar{\tau}_{r\theta c}$ and $\bar{u}_c$ their values given by equations (32) and (34),

$$\frac{\partial \bar{v}_c}{\partial r} - \frac{\bar{v}_c}{r} = \frac{1}{G_{r\theta}} \frac{A_n'}{r^2} \sin n\theta + \frac{1}{E_c} \left(- \frac{n^2 A_n'}{r^2} + \frac{nB_n' \log r}{r} + \frac{C_n'}{r}\right) \sin n\theta$$

Using equation (29) for $\bar{v}_c$,

$$\frac{d\, f_2(r)}{dr} - \frac{f_2(r)}{r} = \left(\frac{1}{G_{r\theta}} - \frac{n^2}{E_c}\right) \frac{A_n'}{r^2} + \frac{nB_n'}{E_c} \frac{\log r}{r} + \frac{1}{E_c} \frac{C_n'}{r}$$

The solution of the above equation is

$$f_2(r) = H_n' r - \left(\frac{1}{2G_{r\theta}} - \frac{n^2}{2E_c}\right) \frac{A_n'}{r} - \frac{nB_n'}{E_c} (1 + \log r) - \frac{C_n'}{E_c}$$

Therefore

$$\bar{v}_c = \left[- \left(\frac{1}{2G_{r\theta}} - \frac{n^2}{2E_c}\right) \frac{A_n'}{r} - \frac{nB_n'}{E_c} (1 + \log r) - \frac{C_n'}{E_c} + H_n' r \right] \sin n\theta \qquad (35)$$

The functional dependence on r of the displacements $\bar{u}_c$ and $\bar{v}_c$ as expressed in equations (34) and (35) having been determined, it is convenient at this point to redefine the constants A_n', B_n', C_n', and H_n'. Equation (34) may be written as follows:

$$\bar{u}_c = (A_n a + B_n \frac{a^2}{r} + C_n a \log r/a) \cos n\theta \qquad (36)$$

Then equation (35) must be replaced by

$$\bar{v}_c = \left[- nA_n a + \left(\frac{1}{2n} \frac{E_c}{G_{r\theta}} - \frac{n}{2}\right) B_n \frac{a^2}{r} - nC_n a (1 + \log r/a) + H_n r \right] \sin n\theta \qquad (37)$$

The displacement equations are thus expressed in terms of the simpler non-dimensional arbitrary constants A_n, B_n, C_n, and H_n. Equations (32) and (33) for the core stresses are expressed in terms of the new constants as follows:

$$\bar{\tau}_{r\theta c} = - \frac{E_c}{n} B_n \frac{a^2}{r^2} \sin n\theta \qquad (38)$$

and

$$q\,(1 + \bar{\epsilon}_\theta)\, a\, d\theta\, dz - (qk + \bar{\sigma}_r)(1 + \bar{\epsilon}_\theta)\, a\, d\theta\, dz$$

$$- \left[qa\,(1 - k) + \bar{N}_\theta \right] (1 + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta} - \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2})\, d\theta\, dz = 0$$

If terms containing products of small quantities, i.e., products of barred quantities, are neglected and the above equation is divided by $d\theta\, dz$, the result may be written as

$$\bar{N}_\theta = - \bar{\sigma}_r a + qa\,(1 - k)\bar{\epsilon}_\theta - qa\,(1 - k)(\frac{1}{a}\frac{\partial \bar{v}}{\partial \theta} - \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2})$$

Since $\bar{\epsilon}_\theta = \frac{\bar{u}}{a} + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta}$, the above equation further reduces to

$$\bar{N}_\theta = - \bar{\sigma}_r a + qa\,(1 - k)(\frac{\bar{u}}{a} + \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2}) \tag{40}$$

The second equilibrium equation of the outer facing is obtained by summing forces in the direction tangent to the element. Thus,

$$\frac{\partial \bar{N}_\theta}{\partial \theta} d\theta\, dz - \bar{\tau}_{r\theta}(1 + \bar{\epsilon}_\theta)\, a\, d\theta\, dz = 0$$

Again by neglecting the term containing the product of the barred quantities and dividing through by $d\theta\, dz$, the above equation is reduced to

$$\frac{\partial \bar{N}_\theta}{\partial \theta} = \bar{\tau}_{r\theta} a \tag{41}$$

The two equilibrium equations which apply to the inner facing are obtained in a manner exactly analogous to that used in obtaining equations (40) and (41). These equations are:

$$\bar{N}'_\theta = \bar{\sigma}'_r b + qak\,(\frac{\bar{u}'}{b} + \frac{1}{b}\frac{\partial^2 \bar{u}'}{\partial \theta^2}) \tag{42}$$

$$\frac{\partial \bar{N}'_\theta}{\partial \theta} = - \bar{\tau}'_{r\theta} b \tag{43}$$

In addition to the four equilibrium equations (40 - 43), the following two equations, based on Hooke's law and the strain-displacement relations, may be written:

$$\bar{\sigma}_{rc} = E_c \left(- B_n \frac{a^2}{r^2} + C_n \frac{a}{r} \right) \cos n\theta \qquad (39)$$

The validity of the above equations is easily verified by the substitution of equations (38) and (39) into the equilibrium equations, equations (24) and (25), and further by the substitution of the expressions given by equations (36 - 39) into the stress-displacement relations, equations (26) and (27). It is of interest to note that the manner in which the core stresses and displacements vary with respect to r is not so simple as is sometimes assumed in problems of this nature.

Equilibrium of Facings

Immediately before buckling, the stress situation in the facings is as illustrated in figure 2. Since $\sigma_r' = (\sigma_{rc})_{r=b}$ and $\sigma_r = (\sigma_{rc})_{r=a}$, the stresses shown in figure 2 can be determined from equations (18), (19), and (20). As stated previously, the stress situation in the facings when the cylinder is in a slightly deformed state is assumed to differ only slightly from the situation which existed just before buckling. In figure 4, which illustrates the differential elements of the deformed facings, the symbols $\bar{N}_\theta$, $\bar{N}_\theta'$, $\bar{\tau}_{r\theta}$, $\bar{\tau}_{r\theta}'$, $\bar{\sigma}_r$, and $\bar{\sigma}_r'$ are used to indicate this small stress variation which has taken place. The assumption of membrane facings eliminates the necessity of considering bending and transverse shear in the individual facings. In formulating the applicable equilibrium equations of the facings, account must be taken of the rotation and stretching of the facing elements during buckling. This effect was found to be negligible in regard to the core. Due to the difference in rotation between the longitudinal elements M and N of the outer facing and the difference in rotation between the longitudinal elements M' and N' of the inner facing, the initial central angle, $d\theta$, becomes $\left(1 + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta} - \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2}\right) d\theta$ and $\left(1 + \frac{1}{b}\frac{\partial \bar{v}'}{\partial \theta} - \frac{1}{b}\frac{\partial^2 \bar{u}'}{\partial \theta^2}\right) d\theta$ for the outer and inner facings, respectively.[4] As a result of the tangential strain in the outer and inner facings, the areas of their differential elements become $(1 + \bar{\epsilon}_\theta)\, a\, d\theta\, dz$ and $(1 + \bar{\epsilon}_\theta')\, b\, d\theta\, dz$, respectively. $\bar{u}$, $\bar{v}$, $\bar{u}'$, $\bar{v}'$, $\bar{\epsilon}_\theta$, and $\bar{\epsilon}_\theta'$ represent small displacements and strains which result when the cylinder goes from the undeformed to the deformed state.

With reference to figure 4 it is seen that two equilibrium equations can be written for each facing. Considering the outer facing first, the summation of forces in the direction normal to the differential element of the outer facing yields

[4] S. Timoshenko. Theory of Elastic Stability, p. 431.

$$\bar{N}_\theta = \frac{Et}{1 - \mu^2} (\bar{\epsilon}_\theta) = \frac{Et}{1 - \mu^2} \left(\frac{\bar{u}}{a} + \frac{1}{a} \frac{\partial \bar{v}}{\partial \theta}\right) \tag{44}$$

$$\bar{N}_\theta' = \frac{Et}{1 - \mu^2} (\bar{\epsilon}_\theta') = \frac{Et}{1 - \mu^2} \left(\frac{\bar{u}'}{b} + \frac{1}{b} \frac{\partial \bar{v}'}{\partial \theta}\right) \tag{45}$$

The following relations enable the right-hand sides of equations (40 - 45) to be expressed in terms of the constants A_n, B_n, C_n, and H_n which appear in the equations for the core displacements, equations (36) and (37):

$$\bar{\sigma}_r = (\bar{\sigma}_{rc})_{r = a} \qquad \text{and} \qquad \bar{\sigma}_r' = (\bar{\sigma}_{rc})_{r = b}$$

or, on the basis of equation (39),

$$\bar{\sigma}_r = E_c \left(- B_n + C_n\right) \cos n\theta \quad \text{and} \quad \bar{\sigma}_r' = E_c \left(- B_n \frac{a^2}{b^2} + C_n \frac{a}{b}\right) \cos n\theta \tag{46}$$

$$\bar{\tau}_{r\theta} = (\bar{\tau}_{r\theta c})_{r = a} \qquad \text{and} \qquad \bar{\tau}_{r\theta}' = (\bar{\tau}_{r\theta c})_{r = b}$$

or, on the basis of equation (38),

$$\bar{\tau}_{r\theta} = - \frac{E_c}{n} B_n \sin n\theta \qquad \text{and} \qquad \bar{\tau}_{r\theta}' = - \frac{E_c}{n} B_n \frac{a^2}{b^2} \sin n\theta \tag{47}$$

$$\bar{u} = (\bar{u}_c)_{r = a} \qquad \text{and} \qquad \bar{u}' = (\bar{u}_c)_{r = b}$$

or, on the basis of equation (36),

$$\bar{u} = (A_n a + B_n a) \cos n\theta \quad \text{and} \quad \bar{u}' = (A_n a + B_n a \frac{a}{b} + C_n a \log b/a) \cos n\theta \tag{48}$$

$$\bar{v} = (\bar{v}_c)_{r = a} \qquad \text{and} \qquad \bar{v}' = (\bar{v}_c)_{r = b}$$

or, on the basis of equation (37),

$$\bar{v} = \left[- n A_n a + \left(\frac{1}{2n} \frac{E_c}{G_{r\theta}} - \frac{n}{2}\right) B_n a - n C_n a + H_n a\right] \sin n\theta$$

and

$$\bar{v}' = \left[- n\, A_n a + \left(\frac{1}{2n}\,\frac{E_c}{G_{r\theta}} - \frac{n}{2}\right) B_n a \left(\frac{a}{b}\right) - n\, C_n a\, (1 + \log b/a) + H_n b \right] \sin n\theta \tag{49}$$

The relationships given by equations (46 - 49) express the assumed continuity conditions on the stresses and displacements at the interfaces. The further assumption that the core extends to the middle surfaces of the facings is implied.

On the basis of the above-listed continuity conditions, equations (40 - 45) can be transformed, respectively, to equations (50 - 55), shown below.

$$\bar{N}_\theta = \left[E_c\, a\, (B_n - C_n) - (n^2 - 1)\, qa\, (1 - k)(A_n + B_n) \right] \cos n\theta \tag{50}$$

$$\frac{\partial \bar{N}_\theta}{\partial \theta} = - \frac{E_c\, a}{n} B_n \sin n\theta \tag{51}$$

$$\bar{N}_\theta' = \left[- E_c\, a\, (B_n \frac{a}{b} - C_n) - (n^2 - 1)\, qak\, \frac{a}{b} \left(A_n + B_n \frac{a}{b} + C_n \log b/a\right) \right] \cos n\theta \tag{52}$$

$$\frac{\partial \bar{N}_\theta'}{\partial \theta} = \frac{E_c\, b}{n} B_n \frac{a^2}{b^2} \sin n\theta \tag{53}$$

$$\bar{N}_\theta = \frac{Et}{1 - \mu^2} \left[- (n^2 - 1)\, A_n + \left(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1\right) B_n - n^2\, C_n + n\, H_n \right] \cos n\theta \tag{54}$$

$$\bar{N}_\theta' = \frac{Et}{1 - \mu^2} \left\{ - (n^2 - 1)\, A_n \frac{a}{b} + \left(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1\right) B_n \frac{a^2}{b^2} - \left[n^2 + (n^2 - 1) \log b/a \right] C_n\, a/b + n\, H_n \right\} \cos n\theta \tag{55}$$

The integration of equations (51) and (53) yields

$$\bar{N}_\theta = \frac{E_c\, a}{n^2} B_n \cos n\theta \tag{56}$$

and

$$\bar{N}_\theta' = - \frac{E_c\, b}{n^2} B_n \frac{a^2}{b^2} \cos n\theta \tag{57}$$

The integration constants are zero, since $\bar{N}_\theta$ and $\bar{N}_\theta'$ are dependent entirely on θ, as shown by equations (50) and (52).

Four independent equations containing the constant A_n, B_n, C_n, and H_n, as well as the load q, can be obtained from equations (50), (52), (54), (55), (56), and (57). If the right-hand sides of equations (50) and (56) are equated, the result is

$$E_c\, a\,(B_n - C_n) - (n^2 - 1)\, qa\,(1 - k)(A_n + B_n) = \frac{E_c\, a}{n^2} B_n$$

The above equation may be written as follows:

$$- (n^2 - 1)(q/E_c)(1 - k)\, A_n - \left[(n^2 - 1)(q/E_c)(1 - k) - \left(\frac{n^2 - 1}{n^2}\right)\right] B_n - C_n = 0 \qquad (58)$$

If the right-hand sides of equations (54) and (56) are equated, the result is

$$\frac{Et}{1 - \mu^2}\left[- (n^2 - 1)\, A_n + \left(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1\right) B_n - n^2\, C_n + n\, H_n\right] = \frac{E_c\, a}{n^2} B_n$$

which reduces to

$$- (n^2 - 1)\, A_n + \left[\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1 - \frac{E_c\, a\,(1 - \mu^2)}{n^2\, Et}\right] B_n - n^2\, C_n + n\, H_n = 0 \qquad (59)$$

If the right-hand side of equations (52) and (57) are equated, the result is

$$- E_c\, a\left(B_n \frac{a}{b} - C_n\right) - (n^2 - 1)\, qak\, \frac{a}{b}\left(A_n + B_n \frac{a}{b} + C_n \log b/a\right) = - \frac{E_c\, a}{n^2} B_n \frac{a}{b}$$

which reduces to

$$(n^2 - 1)(q/E_c)k\, A_n + \left[(n^2 - 1)(q/E_c)k\,\frac{a}{b} + \left(\frac{n^2 - 1}{n^2}\right)\right] B_n$$

$$+ \left[(n^2 - 1)(q/E_c)k \log b/a - \frac{b}{a}\right] C_n = 0 \qquad (60)$$

Finally, if the right-hand sides of equations (55) and (57) are equated, the result is

$$\frac{Et}{1 - \mu^2}\left\{ - (n^2 - 1)\, A_n \frac{a}{b} + \left(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1\right) B_n \frac{a^2}{b^2}\right.$$

$$\left. - \left[n^2 + (n^2 - 1) \log b/a\right] C_n \frac{a}{b} + n\, H_n\right\} = - \frac{E_c\, a}{n^2} B_n \frac{a}{b}$$

which reduces to

$$- (n^2 - 1) \frac{a}{b} A_n + \left[\left(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1\right) \frac{a}{b} + \frac{E_c\, a\, (1 - \mu^2)}{n^2\, Et}\right] \frac{a}{b} B_n$$

$$- \left[n^2 + (n^2 - 1) \log b/a\right] \frac{a}{b} C_n + n\, H_n = 0 \qquad (61)$$

Since H_n appears only in equations (59) and (61), it is eliminated immediately by subtracting equation (59) from equation (61). The result is

$$- (n^2 - 1)\left(\frac{a}{b} - 1\right) A_n + \left[\left(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1\right)\left(\frac{a^2}{b^2} - 1\right) + \frac{E_c\, a\, (1 - \mu^2)}{n^2\, Et}\right.$$

$$\left.\left(\frac{a}{b} + 1\right)\right] B_n - \left[n^2 \left(\frac{a}{b} - 1\right) + (n^2 - 1) \frac{a}{b} \log b/a\right] C_n = 0 \qquad (62)$$

Equations (58), (60), and (62) comprise a system of three equations containing the constants A_n, B_n, and C_n which occur in the expressions for the displacement functions $\bar{u}_c$ and $\bar{v}_c$. A buckled form of equilibrium is possible only if equations (58), (60), and (62) yield non-zero solutions for these constants; this requires that the determinant of the coefficients of A_n, B_n, and C_n be equal to zero. The equation used for the determination of the critical load is obtained from this determinant. Specifically,

$$
\begin{vmatrix}
(n^2 - 1)\, q/E_c\, (1 - k) & (n^2 - 1)\, q/E_c\, (1 - k) - \dfrac{n^2 - 1}{n^2} & 1 \\
(n^2 - 1)(q/E_c)k & (n^2 - 1)(q/E_c)k\, \dfrac{a}{b} + \dfrac{n^2 - 1}{n^2} & (n^2 - 1)(q/E_c)k \log b/a - \dfrac{b}{a} \\
(n^2 - 1)(\dfrac{a}{b} - 1) & - (\dfrac{E_c}{2G_{r\theta}} - \dfrac{n^2}{2} + 1)(\dfrac{a^2}{b^2} - 1) - \dfrac{E_c\, a\, (1 - \mu^2)}{n^2\, Et}(\dfrac{a}{b} + 1) & n^2 (\dfrac{a}{b} - 1) + (n^2 - 1)\dfrac{a}{b} \log b/a
\end{vmatrix} = 0
$$

The following quadratic equation is obtained from the expansion of this determinant:

$$
(q^2/E_c^{\,2})\, k\, (1 - k) \Bigg\{ n^2 \frac{(1 - b/a)^2}{(b/a)^2} + \Bigg[(\frac{E_c}{2G_{r\theta}} + \frac{n^2}{2})(\frac{1 - b^2/a^2}{b^2/a^2})
$$

$$
+ \frac{E_c\, a\, (1 - \mu^2)}{n^2\, Et} (\frac{1 + b/a}{b/a}) \Bigg] \log b/a \Bigg\} + q/E_c \Bigg\{ \frac{(1 - b/a)^2}{(b/a)^2} \Big[b/a
$$

$$
- k\, (1 + b/a) \Big] + \frac{n^2 - 1}{n^2} \Big[1 - k\, (1 - b/a) \Big] \frac{\log b/a}{b/a}
$$

$$
- \frac{1}{n^2 - 1} \Big[b/a + k\, (1 - b/a) \Big] \Bigg[(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1)(\frac{1 - b^2/a^2}{b^2/a^2})
$$

$$
+ \frac{E_c\, a\, (1 - \mu^2)}{n^2\, Et} (\frac{1 + b/a}{b/a}) \Bigg] \Bigg\} - \frac{1}{n^2} \frac{(1 - b/a)^2}{b/a} = 0 \qquad (63)
$$

Analysis of Results

The results of this report are contained in equations (18), (19), and (20), the equations for determining the stresses in a long sandwich cylinder under uniform external loading; and in equation (63), the equation for determining the critical load on a long sandwich cylinder. These equations are repeated below:

$$\sigma_{rc} = q \frac{a}{r} k \tag{18}$$

$$N_\theta = qa (1 - k) \tag{19}$$

$$N_\theta' = qak \tag{20}$$

$$(q^2/E_c{}^2) k (1 - k) \left\{ n^2 \frac{(1 - b/a)^2}{(b/a)^2} + \left[\left(\frac{E_c}{2G_{r\theta}} + \frac{n^2}{2}\right)\left(\frac{1 - b^2/a^2}{b^2/a^2}\right) + \frac{E_c\, a\, (1 - \mu^2)}{n^2\, Et} \left(\frac{1 + b/a}{b/a}\right) \right] \log b/a \right\} + q/E_c \left\{ \frac{(1 - b/a)^2}{(b/a)^2} \left[b/a - k (1 + b/a) \right] + \frac{n^2 - 1}{n^2} \left[1 - k (1 - b/a) \right] \frac{\log b/a}{b/a} - \frac{1}{n^2 - 1} \left[b/a + k (1 - b/a) \right] \left[\left(\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2} + 1\right)\left(\frac{1 - b^2/a^2}{b^2/a^2}\right) + \frac{E_c\, a\, (1 - \mu^2)}{n^2\, Et} \left(\frac{1 + b/a}{b/a}\right) \right] \right\} - \frac{1}{n^2} \frac{(1 - b/a)^2}{b/a} = 0 \tag{63}$$

where, in each of the above equations,

$$k = \frac{1}{(1 + b/a) - \frac{Et \log b/a}{E_c\, a}}$$

and log b/a signifies the natural logarithm of b/a.

The application of equations (18), (19), and (20) to specific examples is very simple. If the ratios, $\frac{E_c\, a}{Et}$ and b/a, are known, the value of k can be determined. Substitution of this value of k into equation (18) yields an expression for the core transverse normal stress in terms of the uniform lateral load, q, and the ratio of the radial distance to the middle surface of the outer facing, a, to the variable radial distance, r. The maximum normal stress in the core always occurs at the inside surface of the core, that is, at $r = b$. Substitution of the k value into equations (19) and (20) yields expressions for the forces per unit length of the facings, N_θ and N_θ', in terms of the load, q, and the radial distance to the middle surface of the outer facing, a. Examination of the expression for k shows that, in general, N_θ and N_θ' are approximately equal since k is approximately equal to one-half in the practical range of dimensions and elastic constants. If the value of the modulus of elasticity of the core approaches infinity, k becomes equal to $\frac{1}{1 + b/a}$; then $N_\theta = q \frac{b}{1 + b/a}$ and $N_\theta' = q \frac{a}{1 + b/a}$. Thus, the statement that N_θ and N_θ' are equal if E_c is infinite and $b/a \approx 1$ is approximately correct and becomes less accurate as the thickness of the cylinder increases. If $E_c \to 0$, $k \to 0$; and the external load is supported entirely by the outer facing. The solution given by equations (18), (19), and (20) may be compared to Reissner's solution of the problem of the closed circular sandwich ring acted upon by a uniform radial load.[5] If desired, equations similar to equations (18), (19), and (20) can be easily obtained for the case of internal pressure, or for a combination of internal and external pressure, by the methods used in this report.

Equation (63), for the determination of the critical load on long sandwich cylinders, is of practical interest only for $n = 2$. A value of $n = 1$ represents, physically, a rigid-body translation of the cylinder, and values of $n > 2$ result in larger values of the critical load than those obtained if $n = 2$. If, in equation (63), the integer 2 is substituted for n and the symbol p is used to represent a uniform lateral pressure, that is, -q, the result is

$$\begin{aligned}
&(p^2/E_c^2)\, k\,(1 - k) \Bigg\{ 4\,\frac{(1 - b/a)^2}{(b/a)^2} + \Bigg[\left(\frac{E_c}{2G_{r\theta}} + 2\right)\left(\frac{1 - b^2/a^2}{b^2/a^2}\right) \\
&\quad + \frac{E_c\, a\,(1 - \mu^2)}{4\,Et} \left(\frac{1 + b/a}{b/a}\right) \Bigg] \log b/a \Bigg\} - p/E_c \Bigg\{ \frac{(1 - b/a)^2}{(b/a)^2} \Big[b/a \\
&\quad - k\,(1 + b/a) \Big] + \frac{3}{4} \Big[1 - k\,(1 - b/a) \Big] \frac{\log b/a}{b/a} - \frac{1}{3} \Big[\frac{b}{a} + k\,(1 - b/a) \Big] \\
&\quad \Bigg[\left(\frac{E_c}{2G_{r\theta}} - 1\right)\left(\frac{1 - b^2/a^2}{b^2/a^2}\right) + \frac{E_c\, a\,(1 - \mu^2)}{4\,Et} \left(\frac{1 + b/a}{b/a}\right) \Bigg] \Bigg\} - \frac{1}{4}\,\frac{(1 - b/a)^2}{b/a} = 0
\end{aligned} \tag{64}$$

[5] National Advisory Committee on Aeronautics Technical Note No. 1832, p. 44.

In the practical range of the values of the dimensions and the physical constants of the sandwich cylinder, equation (64) yields two widely separated positive roots for p, the lower of which represents the critical pressure p_{cr}. Consequently, p_{cr} may be obtained with very good accuracy if the term containing p^2 in equation (64) is neglected.

It is of interest to examine the results given by equation (64) in certain limiting cases. First, it is noted that the critical pressure becomes equal to zero if either E_c or $G_{r\theta}$ is set equal to zero. This is to be expected in view of the assumption of membrane facings. It is of greater interest to examine equation (64) with E_c and $G_{r\theta}$ taken to be infinite. In this case, equation (64) becomes

$$p_{cr} \frac{a (1 - \mu^2)}{Et} = \frac{3 (1 - b/a)^2}{1 + b^2/a^2} \tag{65}$$

The value of p_{cr} obtained from equation (65) should be comparable to the value obtained from the formula for the critical pressure on a long homogeneous cylinder if the moment of inertia used is equal to the moment of inertia of the spaced facings of the sandwich cylinder. The formula for the critical pressure on a long homogeneous thin cylinder is[6]

$$p_{cr} = \frac{3 E I}{r_o^3 (1 - \mu^2)} \tag{66}$$

If the mean radius, r_o, is taken equal to the mean radius of the sandwich cylinder, $\frac{a + b}{2}$, and I is made equal to the moment of inertia of the spaced sandwich cylinder facings, $\frac{t (a - b)^2}{2}$, equation (66) may be written as follows:

$$\frac{p_{cr}\, a (1 - \mu^2)}{Et} = \frac{3 (1 - b/a)^2}{\frac{(1 + b/a)^3}{4}} \tag{67}$$

Comparison of equations (65) and (67) shows that they would be equal if $(1 + \frac{b^2}{a^2})$ were equal to $\frac{(1 + b/a)^3}{4}$. $(1 + b^2/a^2) \approx \frac{(1 + b/a)^3}{4}$ for thin cylinders, i.e., for $\frac{b}{a} \approx 1$. As the b/a ratio decreases from a value of 1,

[6] S. Timoshenko. Theory of Elasticity, p. 216.

equation (67) becomes less accurate since it is derived on the basis of its being a thin cylinder with the pressure applied on the middle surface and thus fails to represent a thicker cylinder with the pressure applied on the outer surface.

Equation (63) may also be applied to problems concerning the stability of long sandwich panels in the form of a portion of a cylinder hinged along the edges $\theta = 0$ and $\theta = 2\alpha$ (fig. 5). If, in equation (63), π/α is substituted for n and -p is substituted for q, the smaller value of p given by equation (63) represents the critical pressure on a panel whose dimensions and properties are known. This solution presupposes the unsymmetrical type of buckling indicated in figure 5, and therefore it may not yield the lowest critical pressure for panels of small curvature, that is, relatively flat panels. A relatively flat sandwich panel may buckle symmetrically with no inflection points between the supports, as in the case of homogeneous panels.[7]

Conclusions

It is felt that the solutions presented in this report are accurate for sandwich cylinders with materials and dimensions such as to render the basic assumptions applicable. The core assumptions are probably sufficiently representative of all practical sandwich construction. The range of applicability of the solutions could be increased somewhat if the effect of the thicknesses of the facings on the overall stiffness of the cylinder were taken into account in the stability analysis. This would entail the use of shell theory rather than membrane theory in regard to the facings. Another extension of the problem which is of greater practical importance would be to obtain the corresponding solutions for cylinders of finite length. A supplementary report containing the solutions for cylinders of finite length is planned for the near future. Numerical computations and curves based on the solutions contained in this report have been omitted since they will appear as limiting cases in the supplementary report.

[7] S. Timoshenko. Theory of Elastic Stability, p. 230.

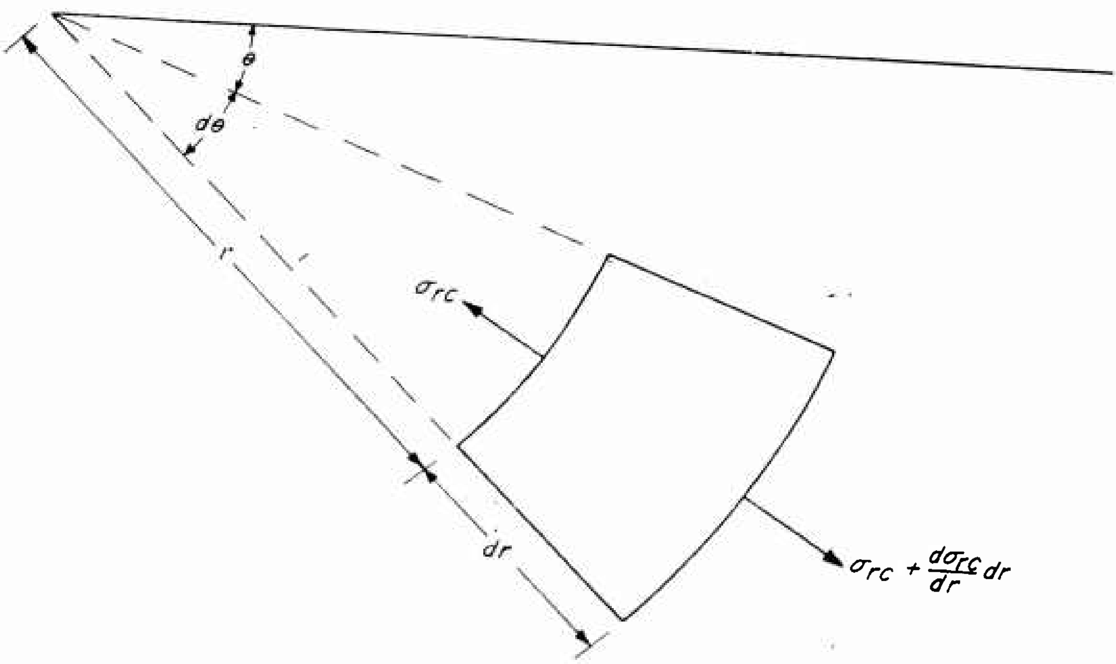

Figure 1.--Differential element of core of uniformly stressed cylinder.

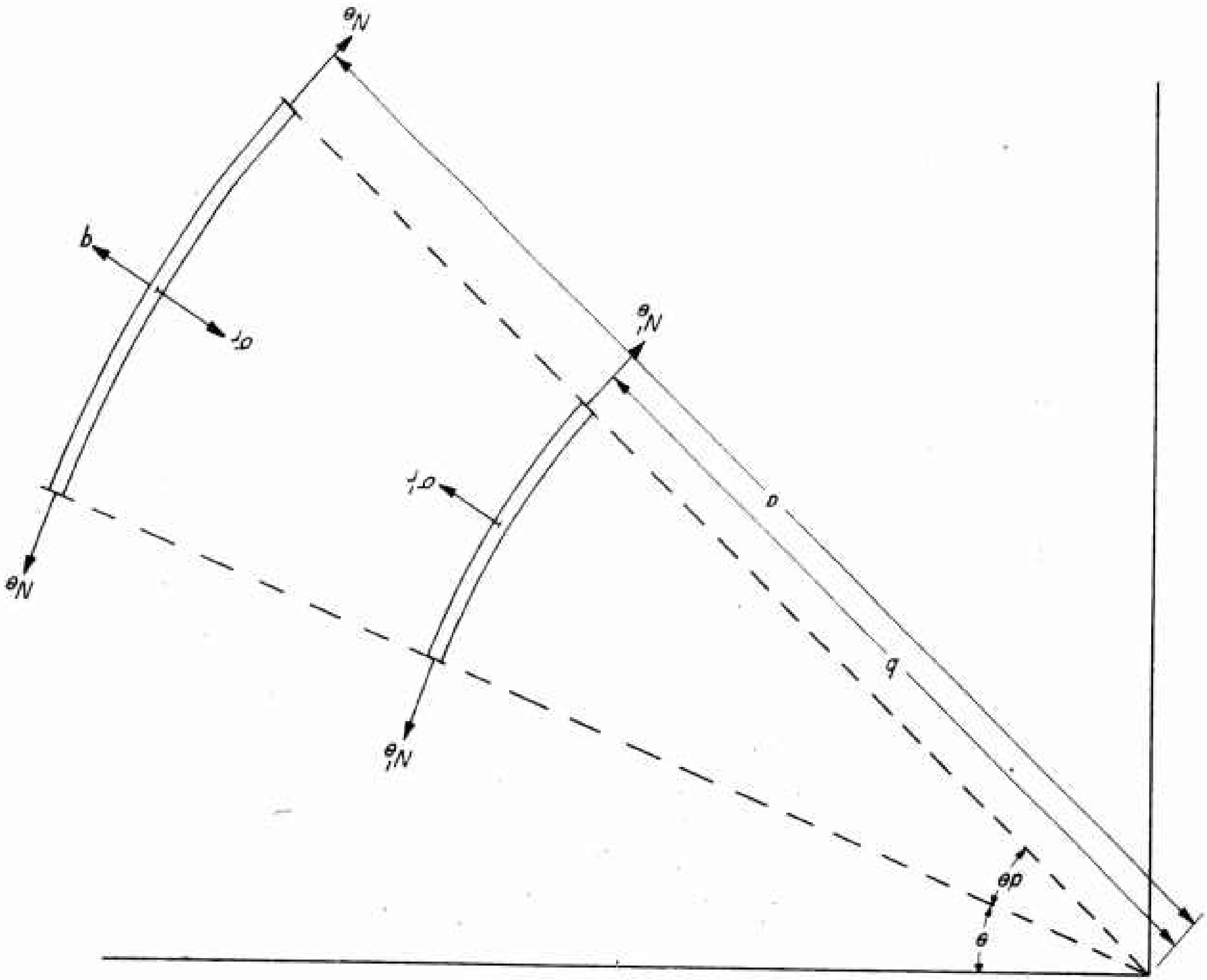

Figure 2.--Differential elements of inner and outer facings of uniformly stressed cylinder.

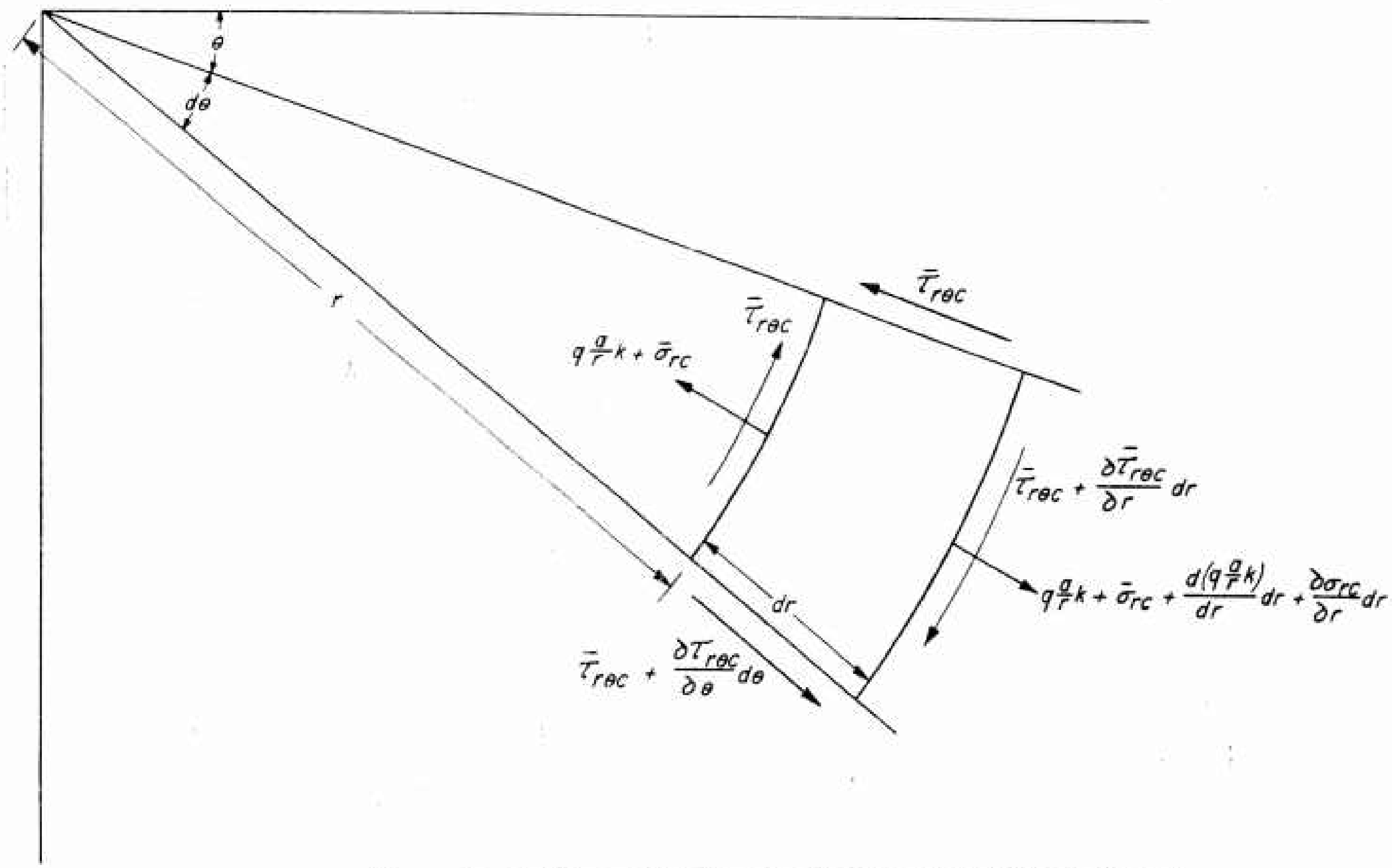

Figure 3.--Differential element of core of slightly deformed cylinder, neglecting changes in geometry of core.

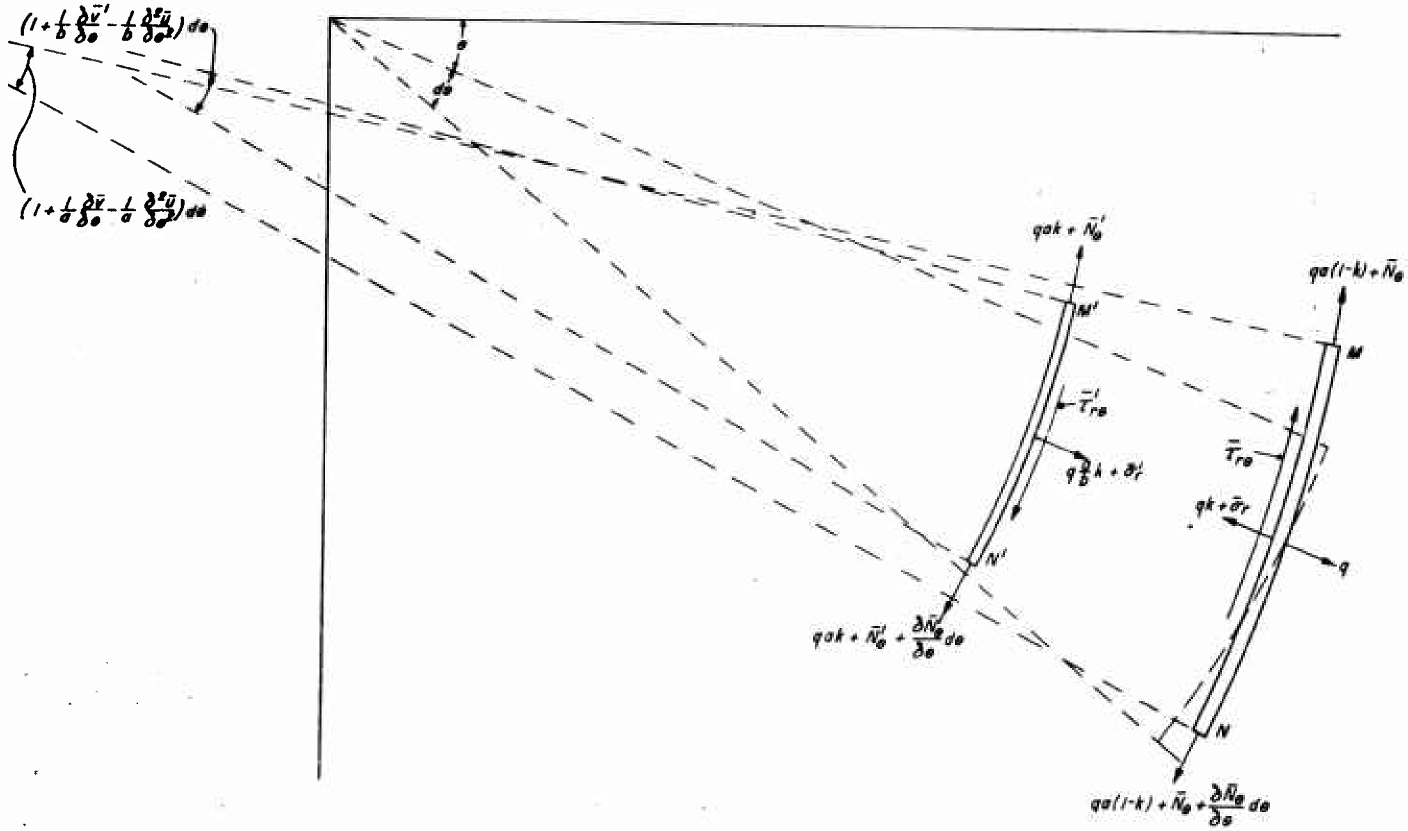

Figure 4.--Differential elements of outer and inner facings of slightly deformed cylinder.

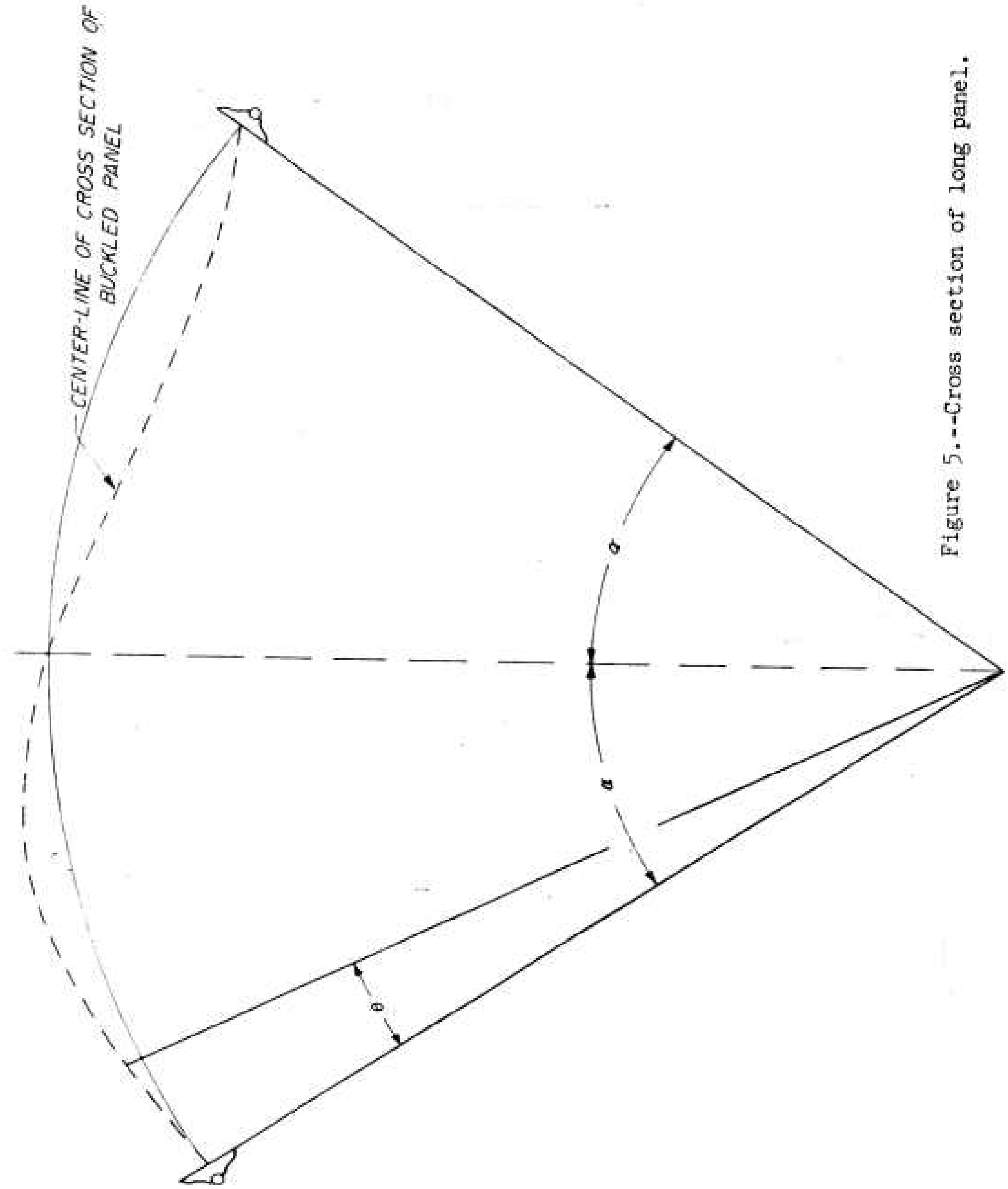

Figure 5.--Cross section of long panel.

SECTION VI

Supplement to

ANALYSIS OF LONG CYLINDERS OF SANDWICH CONSTRUCTION UNDER UNIFORM EXTERNAL LATERAL PRESSURE[1]

Facings of Moderate and Unequal Thicknesses

By

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Summary

A previous analysis of the problem of sandwich cylinders subjected to uniform external lateral loading is extended in order that results may be applied to sandwich cylinders having relatively thick facings of unequal thickness. In the development of the stability criteria, the effect of the stiffness of the individual facings on the stability of the composite cylinder is taken into account. Solutions are obtained from which the stresses and displacements in a stable sandwich cylinder may be determined, and an expression for the determination of the load at which a sandwich cylinder becomes elastically unstable is derived. The sandwich cylinder is assumed to consist of isotropic shell facings and an orthotropic core.

Introduction

This report is a supplement to a previous report that contains a theoretical analysis of sandwich cylinders acted upon by uniform external lateral pressure. Formulas were derived for the stresses in the cylinder, and an expression was obtained from which critical

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[1] This report is one of a series prepared by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics Order No. 01595 and U. S. Air Force Order No. A.F. 18(600)-102.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

pressures may be determined. In that report it was assumed that the facings of the cylinder were thin enough to render membrane theory applicable and that the facings were of equal thickness.

The purpose of this supplementary report is to present solutions for the stresses and for the critical pressures that apply to sandwich cylinders having moderately thick facings of unequal thickness. This requires that the bending moment and the transverse shear in the individual facings be considered in the development of the stability criteria. This extension of the previous work is felt to be of importance in view of the fact that an analysis based on membrane facings may prove inadequate as a design criterion in cases where relatively thick facings are used. It is assumed throughout that buckling takes place at stresses below the elastic limit of the sandwich materials.

The method of analysis used here follows closely the method used in the original report. The same core assumption, namely, that only transverse shear stress and normal stress on planes parallel to the facings are present, is used. It is felt that this assumption applies very well to all practical sandwich constructions because of the relatively low load-carrying capacity in the tangential direction, of the core materials as compared to that of the facing materials. The facings are assumed to be homogeneous and isotropic, and, as indicated previously, they are analyzed on the basis of shell theory rather than membrane theory.

Notation

r, θ	polar coordinates
a	radius to middle surface of outer facing
b	radius to middle surface of inner facing
t_o	thickness of outer facing
t_i	thickness of inner facing
t	thickness of either facing when $t_o = t_i$
E	modulus of elasticity of facings
μ	Poisson's ratio of facings
E_c	modulus of elasticity of core in direction normal to facings
$G_{r\theta}$	modulus of rigidity of core in $r - \theta$ plane

q	intensity of uniform external lateral loading
σ_r, σ_r', σ_{rc}	normal stress in the radial direction acting on the outer facing, on the inner facing, and in the core, respectively
$\bar{\sigma}_r$, $\bar{\sigma}_r'$, $\bar{\sigma}_{rc}$	small normal-stress in the radial direction acting on the outer facing, on the inner facing, and in the core, respectively
$\bar{\tau}_{r\theta}$, $\bar{\tau}_{r\theta}'$	small shear stresses acting on outer and inner facings, respectively
$\bar{\tau}_{r\theta c}$	small transverse shear stress in the core
N_θ, N_θ'	direct stress resultants in the tangential direction in the plane of outer and inner facing, respectively
$\bar{N}_\theta$, $\bar{N}_\theta'$	small direct stress resultants in the tangential direction in the plane of outer and inner facings, respectively
$\bar{M}_\theta$, $\bar{M}_\theta'$	small bending moments per unit length of outer and inner facing, respectively
$\bar{Q}_\theta$, $\bar{Q}_\theta'$	small resultant shear forces per unit length of outer and inner facing, respectively
u, u', u_c	radial displacements of outer facing, inner facing, and core respectively
$\bar{u}$, $\bar{u}'$, $\bar{u}_c$	small radial displacements of outer facing, inner facing, and core, respectively
$\bar{v}$, $\bar{v}'$, $\bar{v}_c$	small tangential displacements of outer facing, inner facing, and core, respectively
ϵ_θ, ϵ_θ'	unit tangential strains in outer and inner facing, respectively
$\bar{\epsilon}_\theta$, $\bar{\epsilon}_\theta'$, $\bar{\epsilon}_{\theta c}$	small unit tangential strains in outer facing, inner facing, and core, respectively
$\bar{\chi}_\theta$, $\bar{\chi}_\theta'$	small changes in curvature of outer and inner facing, respectively
n	number of waves in circumference of cylinder
α	one-half the central angle of curved panel
β	$\dfrac{E_c a\,(1 - \mu^2)}{E t_o}$

γ	$\frac{qa(1 - \mu^2)}{Et_o}$
δ_n	$\frac{E_c}{G_{r\theta}} - \frac{n^2}{2}$
ϕ_o	$\frac{t_o^2}{12a^2}$
ϕ_i	$\frac{t_i^2}{12b^2}$
ψ	$\frac{Et_o(1 - b^2/a^2)}{2G_{r\theta} b (1 - \mu^2)}$
A, B, A_n, B_n, C_n, H_n	arbitrary constants

Stress Analysis

The sandwich cylinder is assumed to be long enough that the effect of the constraints at the ends is negligible. Under a condition of uniform external loading, each cross section remains circular. The dimensions of a cross section of the cylinder and the positive directions of the polar coordinates, r and θ, are indicated in figure 1. In order to avoid any confusion in regard to signs, the intensity of the external load is assumed to act in the positive r direction; obviously, a negative value of q signifies compressive loading. In the analysis which follows, the assumption is made that the core extends to the middle surfaces of the facings and that the load q is applied to the middle surface of the outer facing. This assumption amounts to neglecting the half-thicknesses of the facings as compared to their radii.

Equilibrium of Core

As previously stated, the assumption is made that, in general, the core transmits only normal stresses in the direction perpendicular to the facings and transverse shear stresses. In the case of uniform external loading, it is noted that the transverse shear stress is zero from considerations of symmetry, and the only stress present in the core is the normal stress in the radial direction, σ_{rc}. Considering the equilibrium of the differential element of the core shown in figure 2, the summation of forces in the radial direction results in the following equation:

$$-\sigma_{rc}\, r\, d\theta + \left(\sigma_{rc} + \frac{d\sigma_{rc}}{dr}\right)\left(r + dr\right) d\theta = 0$$

The differential element is considered to be of unit length in the longitudinal direction. The above equilibrium equation reduces to

$$\frac{d\sigma_{rc}}{dr} + \frac{\sigma_{rc}}{r} = 0 \qquad (1)$$

The solution of equation (1) may be written as

$$\sigma_{rc} = E_c \frac{A}{r} \qquad (2)$$

where, for convenience, the constant of integration is represented by $\underline{E_c}\ \underline{A}$. $\underline{E_c}$ represents the modulus of elasticity of the core in the radial direction, and $\underline{A}$ is an arbitrary constant. The radial displacement of the core, $\underline{u_c}$, is related to the radial stress, $\underline{\sigma_{rc}}$, by the following equation:

$$\sigma_{rc} = E_c \frac{du_c}{dr} \qquad (3)$$

From equation (2) it follows that

$$\frac{du_c}{dr} = \frac{A}{r}$$

Integration of the above equation yields

$$u_c = A \log r/a + B \qquad (4)$$

The use of log $\underline{r/a}$ instead of log $\underline{r}$ merely alters the arbitrary constant of integration.

Equilibrium of Facings

With reference to the differential elements of the facings shown in figure 3, it is seen that, when the cylinder is acted upon by an external loading of uniform intensity, $\underline{q}$, the only forces in the facings are the tensile forces per unit length $\underline{N_\theta}$ and $\underline{N_\theta'}$. $\underline{\sigma_r}$ and $\underline{\sigma_r'}$ in figure

3 represent the stresses exerted by the core upon the outer and inner facings, respectively. An equilibrium equation can be obtained for each facing by summing forces in the radial direction on each differential element, considered to be of unit length in the longitudinal direction. The equilibrium equation which pertains to the outer facing is

$$qa\ d\theta - \sigma_r a\ d\theta - N_\theta\ d\theta = 0$$

This equation reduces to

$$N_\theta = a\ (q - \sigma_r)$$

or, since $\sigma_r = (\sigma_{rc})_{r\ =\ a}$

$$N_\theta = a\left[q - (\sigma_{rc})_{r\ =\ a}\right] \tag{5}$$

In a similar manner, the equilibrium equation of the inner facing is found to be

$$N_\theta' = b\ (\sigma_{rc})_{r\ =\ b} \tag{6}$$

From the application of Hooke's law along with the assumption that the tangential stress in the facings is uniformly distributed through their thicknesses, the following relationships are obtained:

$$N_\theta = Et_o\ \epsilon_\theta \tag{7}$$

and

$$N_\theta' = Et_i\ \epsilon_\theta' \tag{8}$$

If the right-hand sides of equations (5) and (7) and the right-hand sides of equations (6) and (8) are equated, the following two equations result:

$$\epsilon_\theta = \frac{a}{Et_o}\left[q - (\sigma_{rc})_{r\ =\ a}\right]$$

and

$$\epsilon_\theta' = \frac{b}{Et_i}\ (\sigma_{rc})_{r\ =\ b}$$

Since $\epsilon_\theta = \frac{u}{a}$ and $\epsilon'_\theta = \frac{u'}{b}$, these two equations become

$$u = \frac{a^2}{Et_o}\left[q - (\sigma_{rc})_{r = a}\right] \qquad (9)$$

and

$$u' = \frac{b^2}{Et_i}(\sigma_{rc})_{r = b} \qquad (10)$$

If continuity of displacements at the interfaces is assumed, then

$$u = (u_c)_{r = a}$$

and

$$u' = (u_c)_{r = b}$$

The above relationships in conjunction with equations (2) and (4) enable equations (9) and (10) to be written as follows:

$$B = \frac{a^2}{Et_o}\left(q - E_c \frac{A}{a}\right) \qquad (11)$$

and

$$A \log b/a + B = \frac{E_c b}{Et_i} A \qquad (12)$$

Equations (11) and (12) may be solved for A and B with the following results:

$$A = \frac{qa}{E_c} k \qquad (13)$$

and

$$B = \frac{qa^2}{Et_o}(1 - k) \qquad (14)$$

where

$$k = \frac{1}{1 + \frac{b}{a}\frac{t_o}{t_i} - \frac{Et_o}{E_c a} \log b/a} \qquad (15)$$

The substitution of the value of A given by equation (13) into equation (2) yields the following expression for the radial stress in the core:

$$\sigma_{rc} = q \frac{a}{r} k \tag{16}$$

When the above value of σ_{rc} is substituted into equations (5) and (6), the following two equations result:

$$N_\theta = qa (1 - k) \tag{17}$$

and

$$N_\theta' = qak \tag{18}$$

The substitution of the values of A and B given by equations (13) and (14) into equation (4) yields the following expression for the core displacement:

$$u_c = \frac{qa^2}{Et_o} [1 - k (1 - \frac{Et_o}{E_c a} \log r/a)] \tag{19}$$

Since $u = (u_c)_{r = a}$ and $u' = (u_c)_{r = b}$, then

$$u = \frac{qa^2}{Et_o} (1 - k) \tag{20}$$

and

$$u' = \frac{qab}{Et_1} k \tag{21}$$

Equations (16) - (21) completely describe the stresses and displacements in the sandwich cylinder subjected to uniform external, lateral loading. In each of these equations the value of k is given by equation (15).

Stability Analysis

In discussing the stability of the sandwich cylinder under uniform external lateral pressure, the equilibrium of a slightly deformed element of the cylinder must be considered. It is assumed that the stress situation that exists in this deformed element differs only slightly from the stress situation that existed just before buckling. The stresses before buckling are given by equations (16), (17), and (18). Since the cross section of the deformed cylinder is no longer circular, the small changes in the stresses and the displacements resulting from buckling will be functions of θ as well as r. Following the system of notation used in the original report, a bar is placed over the appropriate symbol to denote the small stresses, strains, and displacements that occur when the cylinder goes from the initial uniformly stressed circular form to the slightly deformed configuration. Again, it is assumed that the core extends to the middle surfaces of the facings and that the load $\underline{q}$ is applied to the middle surface of the outer facing. This assumption is now somewhat more restrictive than it was in the case of axial symmetry, since it now means that the effect of the interface shear on the bending of the facings and the displacements due to the bending of the individual facings are neglected. However, it is felt that, for cylinders with shell-type facings, the results based on this assumption should be of sufficient accuracy.

Equilibrium of the Core

Since the cylinder is now considered to be slightly deformed, the core is also slightly deformed. A differential element of this core is shown in figure 4. In addition to the radial stress, $\underline{q}\,\frac{a}{r}\,\underline{k}$, given by equation (16), a small radial stress, $\overline{\sigma}_{rc}$, and a small transverse shear stress, $\overline{\tau}_{r\theta c}$, due to buckling must be taken into account. The differential element is considered to be in equilibrium under the action of the stresses shown. Since the small change in the geometry of the core introduces only small terms of higher order into the equilibrium equations, the differential element in figure 4 is shown in its undeformed state. The analysis of the core is exactly the same as that given in the original report, and for this reason only the basic equations will be repeated here.

The equilibrium equations that apply to the core are obtained by summing forces in the radial and tangential directions in figure 4. These equilibrium equations are:

$$\frac{\partial \overline{\sigma}_{rc}}{\partial r} + \frac{\overline{\sigma}_{rc}}{r} + \frac{1}{r}\frac{\partial \overline{\tau}_{r\theta c}}{\partial \theta} = 0 \tag{22}$$

and

$$\frac{\partial \overline{\tau}_{r\theta c}}{\partial r} + \frac{2\overline{\tau}_{r\theta c}}{r} = 0 \tag{23}$$

The following stress-displacement relations are also applicable:

$$\sigma_{rc} = E_c \frac{\partial \overline{u}_c}{\partial r} \tag{24}$$

and

$$\overline{\tau}_{r\theta c} = G_{r\theta}\left[\frac{1}{r}\frac{\partial \overline{u}_c}{\partial \theta} + \frac{\partial \overline{v}_c}{\partial r} - \frac{\overline{v}_c}{r}\right] \tag{25}$$

It was shown in the original report that the small radial and tangential displacements of the core can be completely determined insofar as their dependence on r is concerned. Then, on the assumption that during buckling the circumference of the cylinder subdivides into n waves, the displacement functions are written as follows:

$$\overline{u}_c = \left[A_n + B_n \frac{a}{r} + C_n \log \frac{r}{a}\right] \cos n\theta \tag{26}$$

and

$$\overline{v}_c = \left[- n A_n + \frac{1}{n} \delta_n B_n \frac{a}{r} - n C_n \left(1 + \log \frac{r}{a}\right) + H_n \frac{r}{a}\right] \sin n\theta \tag{27}$$

where

$$\delta_n = \frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$$

and

A_n, B_n, C_n, and H_n are arbitrary constants.

After the substitution of the expressions for $\overline{u}_c$ and $\overline{v}_c$ given by equations (26) and (27) into equations (24) and (25), the expressions for the small core stresses become

$$\overline{\sigma}_{rc} = \frac{E_c}{r}\left[- B_n \frac{a}{r} + C_n\right] \cos n\theta \tag{28}$$

and

$$\overline{\tau}_{r\theta c} = - \frac{1}{n} \frac{E_c}{r} B_n \frac{a}{r} \sin n\theta \tag{29}$$

It may be easily verified that equations (28) and (29) satisfy the equilibrium equations, equations (22) and (23).

Equilibrium of Facings

Figure 5 shows the differential elements of the facings of the slightly deformed cylinder. In addition to the forces that exist just before buckling (given by equations (16), (17), and (18)), the small forces and moments that arise during buckling are shown in the figure. In considering the equilibrium of the facing elements, account is taken of the rotation and stretching of the facings which occurs during buckling. As described in the original report, the initial central angle, $d\theta$, becomes

$$\left(1 + \frac{1}{a} \frac{\partial \overline{v}}{\partial \theta} - \frac{1}{a} \frac{\partial^2 \overline{u}}{\partial \theta^2}\right) d\theta \qquad \text{and} \qquad \left(1 + \frac{1}{b} \frac{\partial \overline{v}'}{\partial \theta} - \frac{1}{b} \frac{\partial^2 \overline{u}'}{\partial \theta^2}\right) d\theta$$

for the outer and inner facings, respectively, and the areas of the differential elements of the outer and inner facings become $(1 + \bar{\epsilon}_\theta)$ a $d\theta$ and $(1 + \bar{\epsilon}'_\theta)$ b $d\theta$, respectively. Three equations of equilibrium can be written for each facing; the differential elements are considered to be of unit length in the longitudinal direction.

Considering first the differential element of the outer facing, the summation of forces in the direction normal to the surface yields

$$q\ (1 + \bar{\epsilon}_\theta)\ a\ d\theta - (qk + \bar{\sigma}_r)(1 + \bar{\epsilon}_\theta)\ a\ d\theta$$

$$- [qa\ (1 - k) + \bar{N}_\theta] \left(1 + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta} - \frac{\partial^2 \bar{u}}{\partial \theta^2}\right) d\theta + \frac{\partial \bar{Q}_\theta}{\partial \theta}\ d\theta = 0$$

If the relationship, $\bar{\epsilon}_\theta = \frac{\bar{u}}{a} + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta}$, is used and small quantities of higher order -- that is, products of barred quantities -- are neglected, the above equation becomes

$$\bar{N}_\theta - \frac{\partial \bar{Q}_\theta}{\partial \theta} = - a\ \bar{\sigma}_r + qa\ (1 - k) \left(\frac{\bar{u}}{a} + \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2}\right) \tag{30}$$

The summation of forces in the tangential direction yields the following equation:

$$- \left[qa\ (1 - k) + \bar{N}_\theta\right] + \left[qa\ (1 - k) + \bar{N}_\theta + \frac{\partial \bar{N}_\theta}{\partial \theta}\ d\theta\right]$$

$$- \bar{\tau}_{r\theta}\ (1 + \bar{\epsilon}_\theta)\ a\ d\theta + \bar{Q}_\theta \left(1 + \frac{1}{a}\frac{\partial \bar{v}}{\partial \theta} - \frac{1}{a}\frac{\partial^2 \bar{u}}{\partial \theta^2}\right) d\theta = 0$$

If small quantities of higher order are again neglected, the above equation may be written as

$$\frac{\partial \bar{N}_\theta}{\partial \theta} + \bar{Q}_\theta = a\ \bar{\tau}_{r\theta} \tag{31}$$

The third equilibrium equation of the outer facing is obtained by equating to zero the summation of moments about point O, shown in figure 5. Thus

$$- \bar{M}_\theta + \left(\bar{M}_\theta + \frac{\partial \bar{M}_\theta}{\partial \theta}\ d\theta\right) + \bar{Q}_\theta\ a\ d\theta = 0$$

The above equation reduces to

$$\frac{\partial \bar{M}_\theta}{\partial \theta} + a\ \bar{Q}_\theta = 0 \tag{32}$$

Similarly the three equilibrium equations which pertain to the inner facing are obtained by summing forces in the normal and tangential directions and by summing moments about the point $\underline{O}'$ using the differential element of the inner facing shown in figure 5. These equilibrium equations are:

$$\overline{N}'_\theta - \frac{\partial \overline{Q}_\theta}{\partial \theta} = b\ \overline{\sigma}'_r + qak\left(\frac{\overline{u}'}{b} + \frac{1}{b}\frac{\partial^2 \overline{u}'}{\partial \theta^2}\right) \tag{33}$$

$$\frac{\partial \overline{N}'_\theta}{\partial \theta} + \overline{Q}'_\theta = -\ b\ \overline{\tau}'_{r\theta} \tag{34}$$

and

$$\frac{\partial \overline{M}'_\theta}{\partial \theta} + b\ \overline{Q}'_\theta = 0 \tag{35}$$

If equation (32) is solved for $\overline{Q}_\theta$ and the resulting value is substituted into equations (30) and (31), the three equilibrium equations of the outer facing are reduced to the following two equations:

$$N_\theta + \frac{1}{a}\frac{\partial^2 \overline{M}_\theta}{\partial \theta^2} = -\ a\ \overline{\sigma}_r + q\ (1 - k)\left(\overline{u} + \frac{\partial^2 \overline{u}}{\partial \theta^2}\right) \tag{36}$$

and

$$\frac{\partial \overline{N}_\theta}{\partial \theta} - \frac{1}{a}\frac{\partial \overline{M}_\theta}{\partial \theta} = a\ \overline{\tau}_{r\theta} \tag{37}$$

In like manner, if equation (35) is solved for $\overline{Q}'_\theta$ and this value is substituted into equations (33) and (34), the three equilibrium equations of the inner facing are reduced to the following two equations:

$$\overline{N}'_\theta + \frac{1}{b}\frac{\partial^2 \overline{M}'_\theta}{\partial \theta^2} = b\ \overline{\sigma}'_r + q\ \frac{a}{b}\ k\left(\overline{u}' + \frac{\partial^2 \overline{u}'}{\partial \theta^2}\right) \tag{38}$$

and

$$\frac{\partial \overline{N}'_\theta}{\partial \theta} - \frac{1}{b}\frac{\partial \overline{M}'_\theta}{\partial \theta} = -\ b\ \overline{\tau}'_{r\theta} \tag{39}$$

From the application of Hooke's law, the following two expressions relating the small tangential forces per unit length, $\underline{\overline{N}_\theta}$ and $\underline{\overline{N}_\theta}$, to the small tangential strains, $\underline{\overline{\epsilon}_\theta}$ and $\underline{\overline{\epsilon}'_\theta}$, are obtained:

$$\overline{N}_\theta = \frac{Et_o}{1 - \mu^2}\,\overline{\epsilon}_\theta \tag{40}$$

and

$$\overline{N}'_\theta = \frac{Et_i}{1 - \mu^2}\,\overline{\epsilon}'_\theta \tag{41}$$

Also, the following two equations relating the bending moments in the facings, $\overline{M}_\theta$ and $\overline{M}'_\theta$, to the changes in curvature in the facings, $\overline{\chi}_\theta$ and $\overline{\chi}'_\theta$, are applicable:[3]

$$\overline{M}_\theta = -\frac{Et_o^3}{12(1 - \mu^2)}\,\overline{\chi}_\theta \tag{42}$$

and

$$\overline{M}'_\theta = -\frac{Et_i^3}{12(1 - \mu^2)}\,\overline{\chi}'_\theta \tag{43}$$

Since[4]

$$\overline{\epsilon}_\theta = \frac{\overline{u}}{a} + \frac{1}{a}\frac{\partial \overline{v}}{\partial \theta}$$

$$\overline{\epsilon}'_\theta = \frac{\overline{u}'}{b} + \frac{1}{b}\frac{\partial \overline{v}'}{\partial \theta}$$

$$\overline{\chi}_\theta = \frac{1}{a^2}\left(\frac{\partial \overline{v}}{\partial \theta} - \frac{\partial^2 \overline{u}'}{\partial \theta^2}\right)$$

and

$$\overline{\chi}'_\theta = \frac{1}{b^2}\left(\frac{\partial \overline{v}'}{\partial \theta} - \frac{\partial^2 \overline{u}'}{\partial \theta^2}\right)$$

equations (40) - (43) may be written as follows:

$$\overline{N}_\theta = \frac{Et_o}{1 - \mu^2}\left(\frac{\overline{u}}{a} + \frac{1}{a}\frac{\partial \overline{v}}{\partial \theta}\right) \tag{44}$$

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[3] Timoshenko, S., and Goodier, J. Theory of Elasticity. New York, 1951.

[4] Timoshenko, S. Theory of Plates and Shells, First Edition. New York 1940.

$$\overline{N}_\theta' = \frac{Et_i}{1 - \mu^2}\left(\frac{\overline{u}'}{b} + \frac{1}{b}\frac{\partial \overline{v}'}{\partial \theta}\right) \tag{45}$$

$$\overline{M}_\theta = - \frac{Et_o^3}{12\ (1 - \mu^2)\ a^2}\left(\frac{\partial \overline{v}}{\partial \theta} - \frac{\partial^2 \overline{u}}{\partial \theta^2}\right) \tag{46}$$

and

$$\overline{M}_\theta' = - \frac{Et_i^3}{12\ (1 - \mu^2)\ b^2}\left(\frac{\partial \overline{v}'}{\partial \theta} - \frac{\partial^2 \overline{u}'}{\partial \theta^2}\right) \tag{47}$$

The requirement is now made that there be continuity of displacements at the interfaces; that is,

$$\overline{u} = (\overline{u}_c)_{r = a} \tag{48}$$

$$\overline{u}' = (\overline{u}_c)_{r = b} \tag{49}$$

$$\overline{v} = (\overline{v}_c)_{r = a} \tag{50}$$

and

$$\overline{v}' = (\overline{v}_c)_{r = b} \tag{51}$$

The use of equations (48) - (51) enables equations (44) - (47) to be written as follows:

$$\overline{N}_\theta = \frac{Et_o}{a\ (1 - \mu^2)}\left(\overline{u}_c + \frac{\partial \overline{v}_c}{\partial \theta}\right)_{r = a} \tag{52}$$

$$\overline{N}_\theta' = \frac{Et_i}{b\ (1 - \mu^2)}\left(\overline{u}_c + \frac{\partial \overline{v}_c}{\partial \theta}\right)_{r = b} \tag{53}$$

$$\overline{M}_\theta = - \frac{Et_o^3}{12a^2\ (1 - \mu^2)}\left(\frac{\partial \overline{v}_c}{\partial \theta} - \frac{\partial^2 \overline{u}_c}{\partial \theta^2}\right)_{r = a} \tag{54}$$

and

$$\overline{M}_\theta' = - \frac{Et_i^3}{12b^2\ (1 - \mu^2)}\left(\frac{\partial \overline{v}_c}{\partial \theta} - \frac{\partial^2 \overline{u}_c}{\partial \theta^2}\right)_{r = b} \tag{55}$$

With the aid of equations (48) - (55) and the fact that

$$\overline{\tau}_{r\theta} = (\overline{\tau}_{r\theta c})_{r = a} \tag{56}$$

$$\overline{\tau}'_{r\theta} = (\overline{\tau}_{r\theta c})_{r = b} \tag{57}$$

$$\overline{\sigma}_r = (\overline{\sigma}_{rc})_{r = a} \tag{58}$$

and

$$\overline{\sigma}'_r = (\overline{\sigma}_{rc})_{r = b} \tag{59}$$

the equilibrium equations of the facings, equations (36) - (39), may be expressed entirely in terms of the core displacements and stresses. For example, if the right hand sides of equations (52), (54), (56), (58), and (48) are substituted for $\overline{N}_\theta$, $\overline{M}_\theta$, $\overline{\tau}_{r\theta}$, $\overline{\sigma}_r$, and $\overline{u}$, respectively, in equilibrium equation (36), the result is

$$\frac{Et_o}{a\ (1 - \mu^2)} \left(\overline{u}_c + \frac{\partial \overline{v}_c}{\partial \theta}\right)_{r = a} - \frac{Et_o{}^3}{12\ (1 - \mu^2)\ a^3} \left(\frac{\partial^3 \overline{v}_c}{\partial \theta^3} - \frac{\partial^4 \overline{u}_c}{\partial \theta^4}\right)_{r = a}$$

$$= - a\ (\overline{\sigma}_{rc})_{r = a} + q\ (1 - k) \left(\overline{u}_c + \frac{\partial^2 \overline{u}_c}{\partial \theta^2}\right)_{r = a} \tag{60}$$

In a similar manner, equations (37) - (39) may be transformed, respectively, to the following three equations:

$$\frac{Et_o}{a\ (1 - \mu^2)} \left(\frac{\partial \overline{u}_c}{\partial \theta} + \frac{\partial^2 \overline{v}_c}{\partial \theta^2}\right)_{r = a} + \frac{Et_o{}^3}{12\ (1 - \mu^2)\ a^3} \left(\frac{\partial^2 \overline{v}_c}{\partial \theta^2} - \frac{\partial^3 \overline{u}_c}{\partial \theta^3}\right)_{r = a}$$

$$= a\ (\tau_{r\theta c})_{r = a} \tag{61}$$

$$\frac{Et_i}{b\ (1 - \mu^2)} \left(\overline{u}_c + \frac{\partial \overline{v}_c}{\partial \theta}\right)_{r = b} - \frac{Et_i{}^3}{12\ (1 - \mu^2)\ b^3} \left(\frac{\partial^3 \overline{v}_c}{\partial \theta^3} - \frac{\partial^4 \overline{u}_c}{\partial \theta^4}\right)_{r = b}$$

$$= b\ (\overline{\sigma}_{rc})_{r = b} + q\ \frac{a}{b}\ k \left(\overline{u}_c + \frac{\partial^2 \overline{u}_c}{\partial \theta^2}\right)_{r = b} \tag{62}$$

and

$$\frac{Et_i}{b\,(1-\mu^2)}\left(\frac{\partial \bar{u}_c}{\partial \theta} + \frac{\partial^2 \bar{v}_c}{\partial \theta^2}\right)_{r=b} + \frac{Et_i^3}{12\,(1-\mu^2)\,b^3}\left(\frac{\partial^2 \bar{v}_c}{\partial \theta^2} - \frac{\partial^3 \bar{u}_c}{\partial \theta^3}\right)_{r=b}$$

$$= -\,b\,(\bar{\tau}_{r\theta c})_{r=b} \tag{63}$$

If the expressions given by equations (26) - (29) for the core displacements and stresses are substituted into equations (60) - (63), four equations containing the parameters $\underline{A_n}$, $\underline{B_n}$, $\underline{C_n}$, and $\underline{H_n}$ are obtained. These four equations are:

$$[(n^2-1)\,\gamma\,(1-k) - (n^2-1)]\,A_n + [(n^2-1)\,\gamma\,(1-k) + \delta_n\,(1+n^2\,\phi_o) + (1+n^4\,\phi_o) - \beta]\,B_n + [\beta - n^2\,(1+n^2\,\phi_o)]\,C_n + [n\,(1+n^2\,\phi_o)]\,H_n = 0 \tag{64}$$

$$(n^2-1)\,A_n + [-\delta_n\,(1+\phi_o) - (1+n^2\,\phi_o) + \frac{\beta}{n^2}]\,B_n + [n^2\,(1+\phi_o)]\,C_n + [-n\,(1+\phi_o)]\,H_n = 0 \tag{65}$$

$$[(n^2-1)\,\gamma k\,\frac{t_o}{t_i} - (n^2-1)]\,A_n + [(n^2-1)\,\frac{a}{b}\,\gamma k\,\frac{t_o}{t_i} + \delta_n\,\frac{a}{b}(1+n^2\,\phi_i) + \frac{a}{b}\,(1+n^4\,\phi_i) + \beta\,\frac{t_o}{t_i}]\,B_n + [(n^2-1)\,\gamma k\,\frac{t_o}{t_i}\,\log b/a - \beta\,\frac{b}{a}\,\frac{t_o}{t_i} - (n^2-1)\,\log b/a - n^2\,(1+n^2\,\phi_i)]\,C_n + [n\,b/a\,(1+n^2\,\phi_i)]\,H_n = 0 \tag{66}$$

and

$$(n^2-1)\,A_n + [-\delta_n\,\frac{a}{b}\,(1+\phi_i) - \frac{a}{b}\,(1+n^2\,\phi_i) - \frac{\beta}{n^2}\,\frac{t_o}{t_i}]\,B_n + [(n^2-1)\,\log b/a + n^2\,(1+\phi_i)]\,C_n + [-n\,b/a\,(1+\phi_i)]\,H_n = 0 \tag{67}$$

where, in each of the above equations,

$$\gamma = \frac{qa\,(1-\mu^2)}{Et_o}$$

$$\delta_n = \frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$$

$$\phi_o = \frac{t_o^2}{12a^2}$$

$$\phi_i = \frac{t_i^2}{12b^2}$$

and

$$\beta = \frac{E_c a\ (1 - \mu^2)}{Et_o}$$

Each of the terms in equations (64) - (67) contains one of the parameters $\overline{A_n}$, $\overline{B_n}$, $\overline{C_n}$, and $\overline{H_n}$ that appear in the displacement equations of the cylinder. A buckled form of equilibrium is possible only if equations (64) - (67) yield solutions for these parameters which are different from zero. This requires that the determinant of the coefficients of these parameters must be equal to zero. This determinant may be written as follows:

$$
\begin{vmatrix}
\gamma (1 - k) & \gamma (1 - k) + (\delta_n + n^2) \phi_o - \frac{\beta}{n^2} & \frac{\beta}{n^2 - 1} - n^2 \phi_o & \phi_o \\
n^2 - 1 & - \delta_n (1 + \phi_o) - (1 + n^2 \phi_o) + \frac{\beta}{n^2} & n^2 (1 + \phi_o) & - (1 + \phi_o) \\
\gamma k \frac{t_o}{t_i} & \gamma k \frac{a}{b} \frac{t_o}{t_i} + (\delta_n + n^2) \phi_i \frac{a}{b} + \frac{\beta}{n^2} \frac{t_o}{t_i} & \gamma k \frac{t_o}{t_i} \log b/a - \frac{\beta}{n^2 - 1} \frac{b}{a} \frac{t_o}{t_i} - n^2 \phi_i & \phi_i \, b/a \\
n^2 - 1 & - \delta_n \frac{a}{b} (1 + \phi_i) - \frac{a}{b} (1 + n^2 \phi_i) - \frac{\beta}{n^2} \frac{t_o}{t_i} & (n^2 - 1) \log b/a + n^2 (1 + \phi_i) & - b/a (1 + \phi_i)
\end{vmatrix} = 0 \qquad (68)
$$

Since terms containing γ appear in only two of the four rows, the expansion of the determinant shown above results in a quadratic equation in γ. The two values of γ that satisfy the quadratic equation are, in general, widely separated negative roots. The root that is the lower in absolute value is proportional to the critical load on the cylinder and is called γ_{cr}. After γ_{cr} has been determined, the critical load is obtained in accordance with the definition of γ previously given; that is,

$$q_{cr} = \frac{Et_o}{a(1 - \mu^2)} \gamma_{cr}$$

The physical significance of the fact that the determinant has two eigenvalues will be discussed later.

Analysis of Results

The results of this report are contained in equations (16) through (21), the equations for determining the stresses and displacements in the stable sandwich cylinder, and in equation (68), the determinant from which the critical load on a sandwich cylinder may be obtained. These results apply to long sandwich cylinders that have thin shell facings and are subjected to uniform external, lateral loading.

The use of equations (16) through (21) is self-explanatory; if the dimensions and material properties of a given sandwich cylinder are known, the stresses and displacements in terms of the external load, q, may be easily computed. In the case of sandwich cylinders having facings of equal thickness, equations (16) - (21) reduce to equations (18) - (23) of the original report. It is of interest to compare equations (16) - (21) with results obtained by Reissner.[5] If Reissner's equations are expressed in the notation used here, the following equations for the cylinder stresses result:

$$N_\theta = qa\left[\frac{1 + \frac{Et}{E_c a}(1 - b/a)}{2 + \frac{Et}{E_c a}(1 - b/a)}\right]$$

$$N_\theta' = qa\left[\frac{1}{2 + \frac{Et}{E_c a}(1 - b/a)}\right]$$

and

$$\sigma_{rc}\ (\text{at middle surface of core}) = q\left(\frac{2}{1 + b/a}\right)\left[\frac{1}{2 + \frac{Et}{E_c a}(1 - b/a)}\right].$$

Equations (16) - (18), for facings of equal thickness, may be written as follows:

$$N_\theta = qa\left[\frac{\frac{b}{a} - \frac{Et}{E_c a}\log b/a}{1 + b/a - \frac{Et}{E_c a}\log b/a}\right]$$

$$N_\theta' = qa\left[\frac{1}{1 + b/a - \frac{Et}{E_c a}\log b/a}\right]$$

and

$$\sigma_{rc}\ (\text{at middle surface of core}) = q\left(\frac{2}{1 + b/a}\right)\left[\frac{1}{1 + b/a - \frac{Et}{E_c a}\log b/a}\right]$$

Since the first term in the series expansion of log $\underline{b/a}$ is - (1 - b/a), it may be noted that, for cylinders with b/a ratios close to 1, the equations of Reissner yield results that are very nearly the same as those given by equations (16), (17), and (18). Reissner's results are based on the assumption that a - b << a and hence become increasingly less accurate as the cylinder thickness is increased.

[5] Reissner, Eric, Small Bending and Stretching of Sandwich-type Shells. National Advisory Committee on Aeronautics, Tech. Note 1832. 1949.

For the determination of critical loads on long-sandwich cylinders, equation (68) with $n = 2$ is used. The case $n = 1$ represents rigid body translation of the cylinder, and values of $n > 2$ result in higher critical loads than that obtained with $n = 2$. With $n = 2$, equation (68) becomes

$$\begin{vmatrix} \gamma (1 - k) & \gamma (1 - k) + (\delta_2 + 4)\, \phi_o - \frac{\beta}{4} & \frac{\beta}{3} - 4\phi_o & \phi_o \\ 3 & -\delta_2 (1 + \phi_o) - (1 + 4\phi_o) + \frac{\beta}{4} & 4 (1 + \phi_o) & -(1 + \phi_o) \\ \gamma k \frac{t_o}{t_i} & \gamma k \frac{a}{b} \frac{t_o}{t_i} + (\delta_2 + 4)\, \phi_i \frac{a}{b} + \frac{\beta}{4} \frac{t_o}{t_i} & \gamma k \frac{t_o}{t_i} \log b/a - \frac{\beta}{3} \frac{b}{a} \frac{t_o}{t_i} - 4\phi_i & \phi_i \frac{b}{a} \\ 3 & -\delta_2 \frac{a}{b} (1 + \phi_i) - \frac{a}{b} (1 + 4\phi_i) - \frac{\beta}{4} \frac{t_o}{t_i} & 3 \log b/a + 4 (1 + \phi_i) & -\frac{b}{a} (1 + \phi_i) \end{vmatrix} = 0 \qquad (69)$$

Since a literal expansion of the above determinant results in very little simplification, equation (69) is left in this form. Numerical solutions can be obtained quite readily for specific cases. If, in equation (69), ϕ_o and ϕ_i are set equal to zero and $t_o = t_i$, the determinant reduces to that obtained on the assumption of membrane facings presented in the original report.

The two eigenvalues of the determinant correspond to the two configurations shown in figure 6. Obviously, the critical load that corresponds to the configuration shown in figure 6 (b) is considerably higher than that which corresponds to 6 (a) and is of no practical interest in the long sandwich cylinder problem.

A simpler expression for the determination of critical loads is obtained if the modulus of elasticity of the core in the radial direction is assumed to be infinite. Under this assumption, equation (68) becomes

$$\gamma_{cr} = -(n^2 - 1)\left(\frac{1 + \frac{b}{a}\frac{t_o}{t_i}}{1 + \frac{b^2}{a^2}\frac{t_o}{t_i}}\right)\left\{\frac{\left(1 - \frac{b}{a}\right)^2 + \left(\phi_o \frac{t_o}{t_i} + \phi_i \frac{b}{a}\right)\left(\frac{t_i}{t_o} + \frac{b}{a}\right) + n^2 \psi \left[\phi_o (1 + \phi_i) \frac{b}{a} + \phi_i (1 + \phi_o) \frac{t_i}{t_o}\right]}{(1 + \phi_o)\frac{t_o}{t_i} + (1 + \phi_i)\frac{b}{a} + n^2 \psi (1 + \phi_o)(1 + \phi_i)}\right\} \qquad (70)$$

where

$$\psi = \frac{Et_o \left(1 - \frac{b^2}{a^2}\right)}{2G_{r\theta}\, b\, (1 - \mu^2)}$$

Equation (70), with $n = 2$, yields values of γ_{cr} within 3 percent of the values obtained from equation (69) for usual sandwich constructions.

For cylinders having very thin facings (membrane facings) of equal thickness, ϕ_o and ϕ_i are assumed to be zero, and equation (70) reduces to

$$\gamma_{cr} = -(n^2 - 1)\frac{\left(1 - \frac{b}{a}\right)^2}{\left(1 + \frac{b^2}{a^2}\right)\left[1 + \frac{n^2\, Et\, \left(1 - \frac{b}{a}\right)}{2G_{r\theta}\, b\, (1 - \mu^2)}\right]} \tag{71}$$

Equation (71), for $n = 2$, becomes

$$\gamma_{cr} = -\frac{3\left(1 - \frac{b}{a}\right)^2}{\left(1 + \frac{b^2}{a^2}\right)\left[1 + \frac{2Et\, \left(1 - \frac{b}{a}\right)}{G_{r\theta}\, b\, (1 - \mu^2)}\right]} \tag{72}$$

The value of q_{cr} is then determined from the definition previously given,

$$q_{cr} = \frac{Et}{a\,(1 - \mu^2)}\,\gamma_{cr}$$

Equation (72) may be used as a good approximation to the expression obtained in the original report (equation 64) for the critical load on cylinders with membrane facings.

It is of interest to examine the results given by equation (68) for certain limiting cases other than that of membrane facings. Some of these results are given in table 1. In each of these limiting cases, ϕ_o and ϕ_i are neglected as compared to 1.

Equation (68) may be used for determining the approximate value of the critical load on long sandwich panels in the form of a portion of a cylinder hinged along the edges $\theta = 0$ and $\theta = 2\alpha$ as shown in figure (7). If, in equation (68), π/α is substituted for n, the smaller absolute value of q obtained from equation (68) represents the critical load on a panel whose dimensions and properties are known. This solution applies to the unsymmetrical type of buckling shown in figure (7). If, as in the case of relatively flat panels, symmetrical buckling with no inflection point between the supports occurs, equation (68) is not applicable.

Conclusions

The purpose of this report was to extend the previous work done in connection with sandwich cylinders subjected to uniform lateral loading by taking into account the effect of the stiffnesses of the individual facings of the cylinder and also by making the results applicable to sandwich cylinders having facings of unequal thickness. The results indicate that for the majority of sandwich cylinders the analysis based on membrane facings is sufficiently accurate. However, for cylinders having relatively thick facings, say in the neighborhood of one-fourth of the core thickness, the facing stiffnesses have an appreciable effect on the critical load on the composite cylinder. Since the magnitude of the effect of the facing stiffnesses is dependent not only on the dimensions of the cylinder but also on the mechanical properties of the core and facing materials, equation (68) of this report should be used for computing the critical load if indications are that this effect may be important. In all cases of sandwich cylinders having facings of unequal thickness, the equations of this report should be used for computing stresses prior to buckling and for the determination of critical loads.

Table 1.--Results obtained with equation 68 for certain limiting cases

E_c	$G_{r\theta}$	t_o	t_i	q_{cr}
E_c	$G_{r\theta}$	o	t_i	$-\dfrac{Et_i^3}{4(1-\mu^2)b^3}\left(\dfrac{b}{a}\right)$
E_c	$G_{r\theta}$	t_o	o	$-\dfrac{Et_o^3}{4(1-\mu^2)a^3}$
0	$G_{r\theta}$	t_o	t_i	$-\dfrac{Et_o^3}{4(1-\mu^2)a^3}$
∞	0	t_o	t_i	$-3\dfrac{Et_o}{a(1-\mu^2)}\left(\dfrac{1+\dfrac{b}{a}\dfrac{t_o}{t_i}}{1+\dfrac{b^2}{a^2}\dfrac{t_o}{t_i}}\right)\left(\phi_o\dfrac{b}{a}+\phi_i\dfrac{t_i}{t_o}\right)$
∞	∞	t	t	$-3\dfrac{Et}{a(1-\mu^2)}\left[\dfrac{\left(1-\dfrac{b}{a}\right)^2+\dfrac{t^2}{12ab}\left(1+\dfrac{b}{a}\right)^2}{1+\dfrac{b^2}{a^2}}\right]$

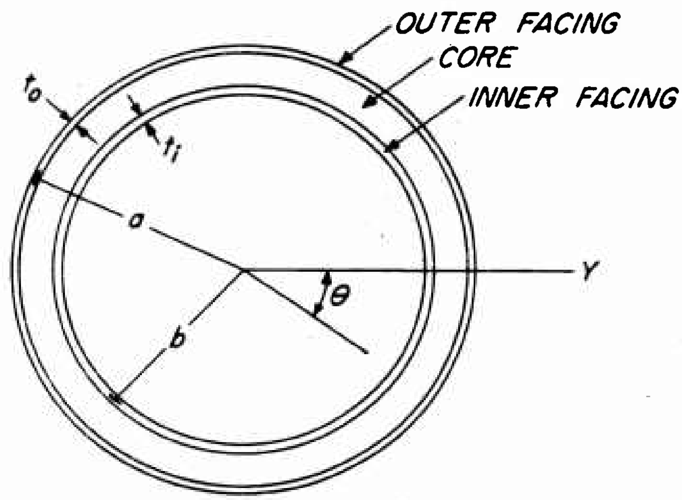

Figure 1.--Cross section of sandwich cylinder.

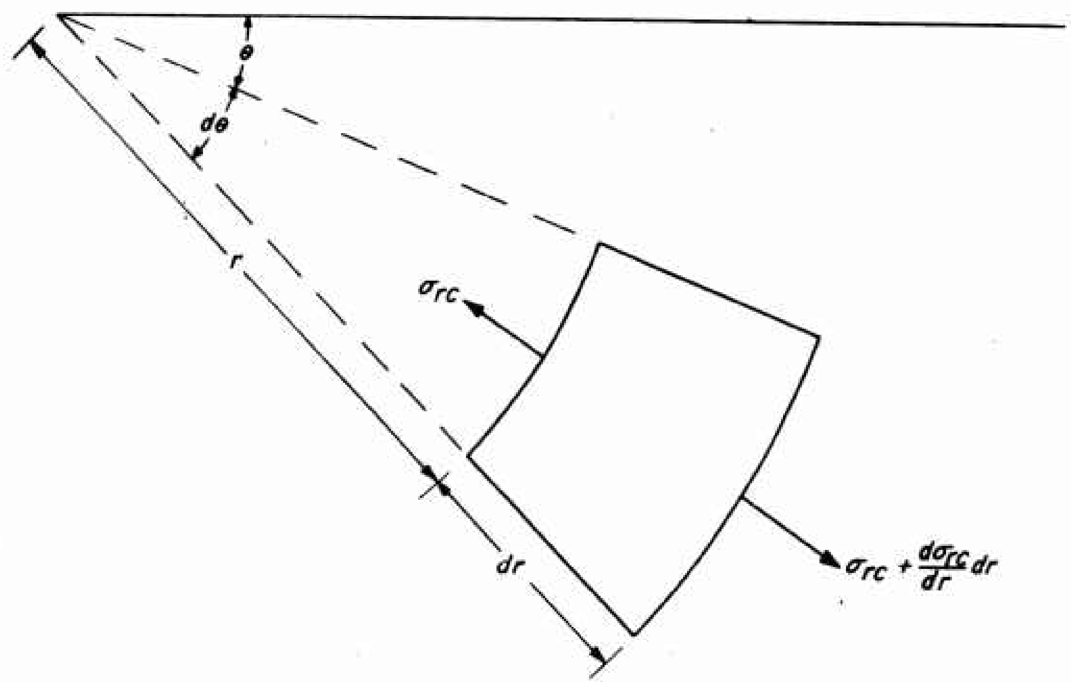

Figure 2.--Differential element of core of uniformly stressed cylinder.

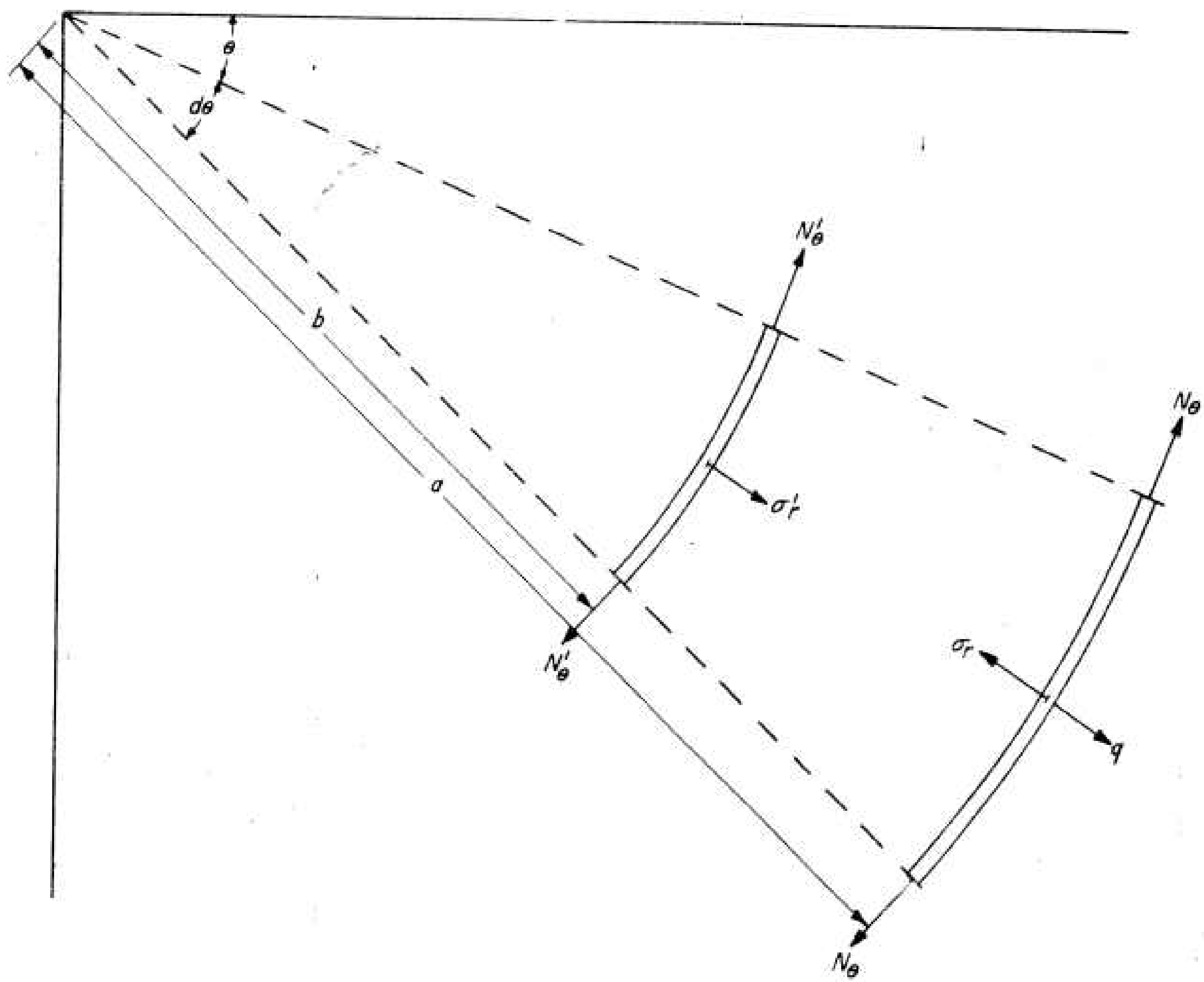

Figure 3.--Differential elements of inner and outer facings of uniformly stressed cylinder.

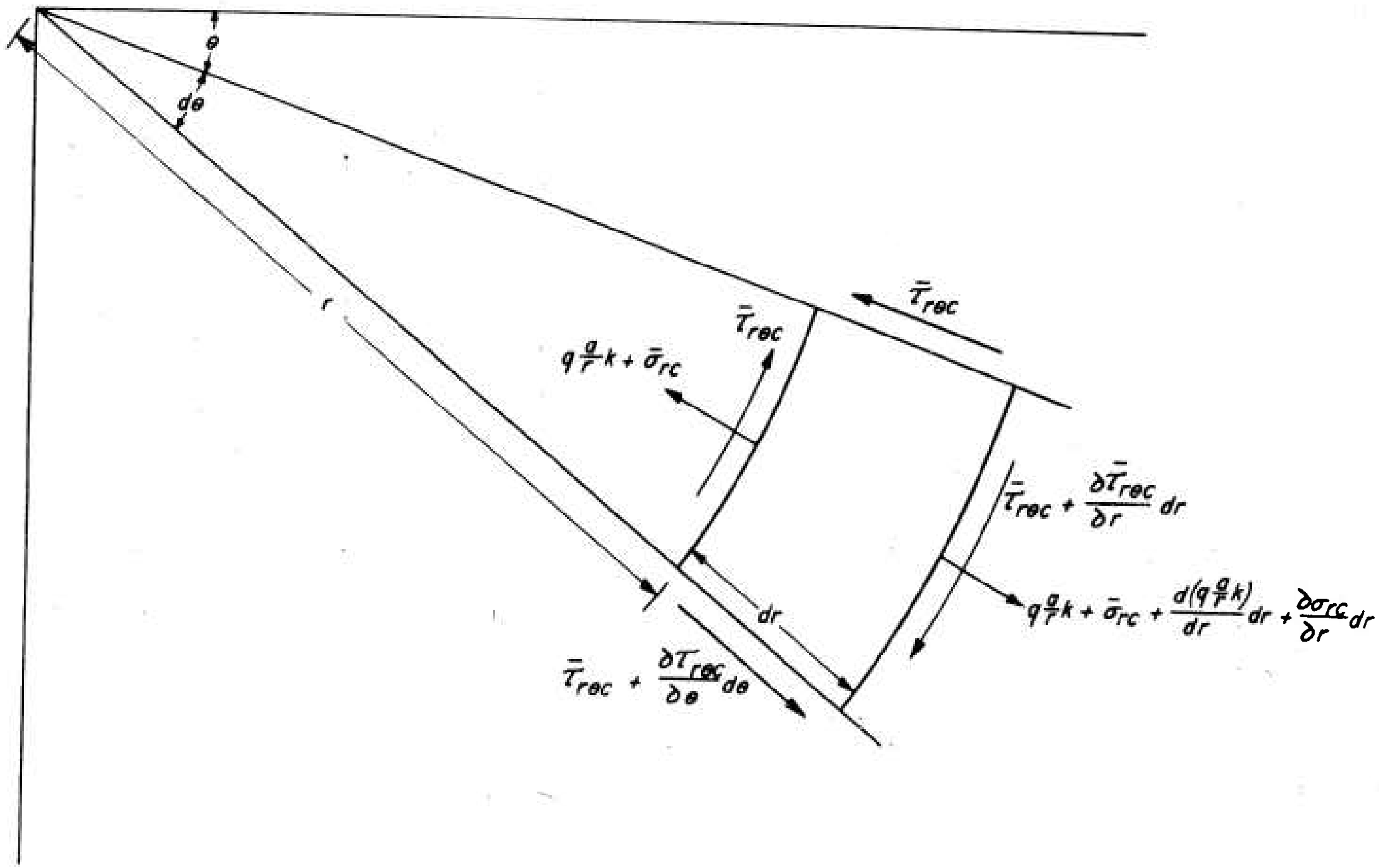

Figure 4.--Differential element of core of slightly deformed cylinder, neglecting changes in geometry of core.

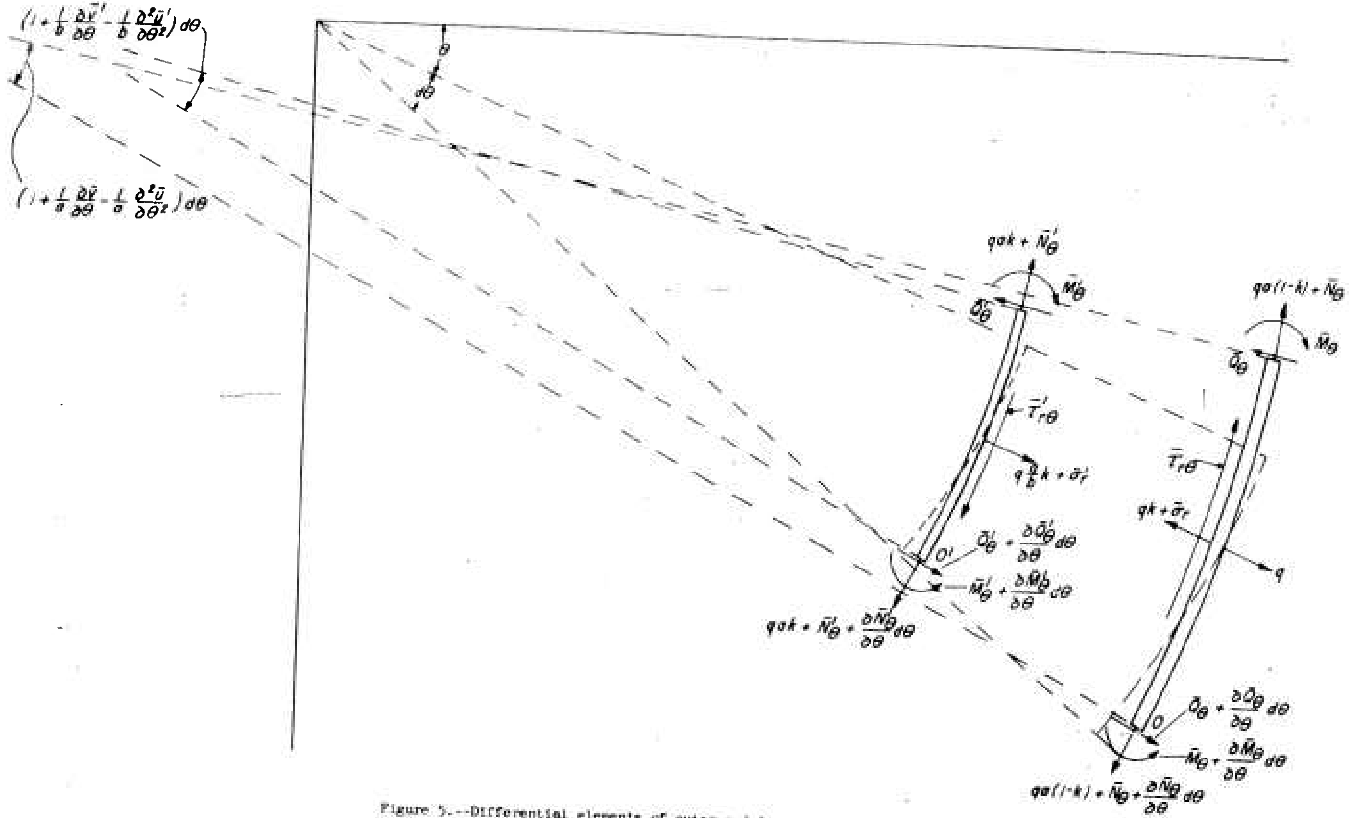

Figure 5.--Differential elements of outer and inner facing of slightly deformed cylinder.

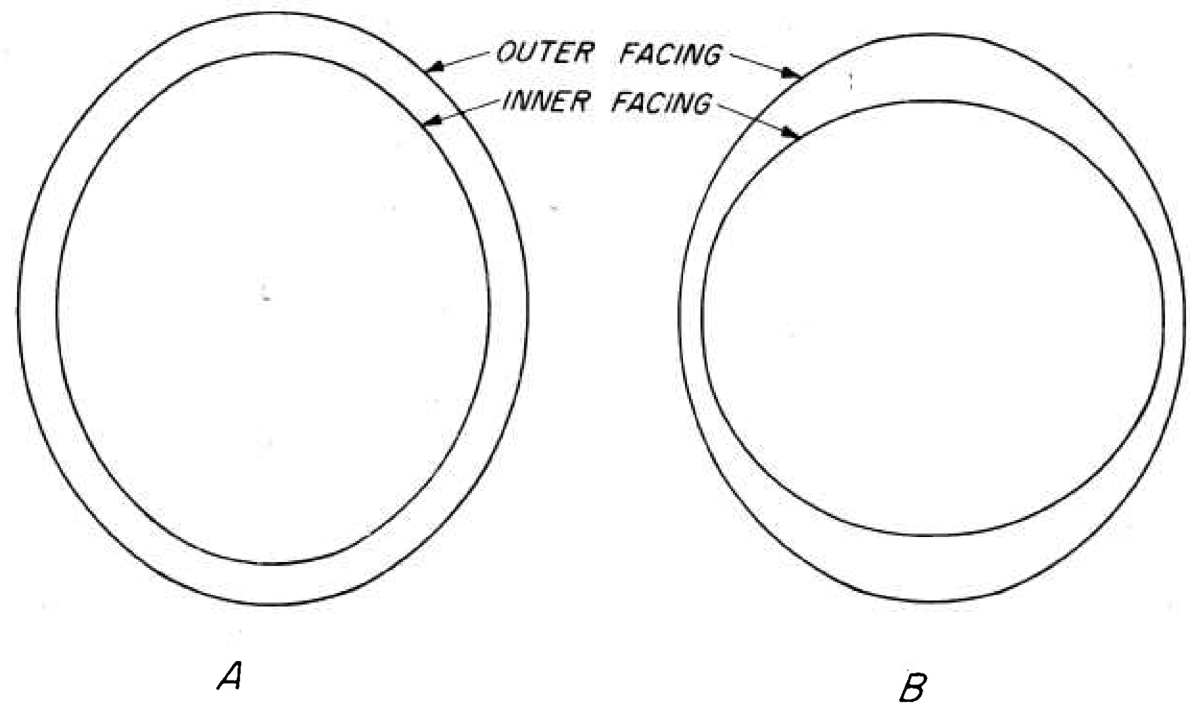

Figure 6.--Configurations corresponding to the two eigenvalues of the stability determinant.

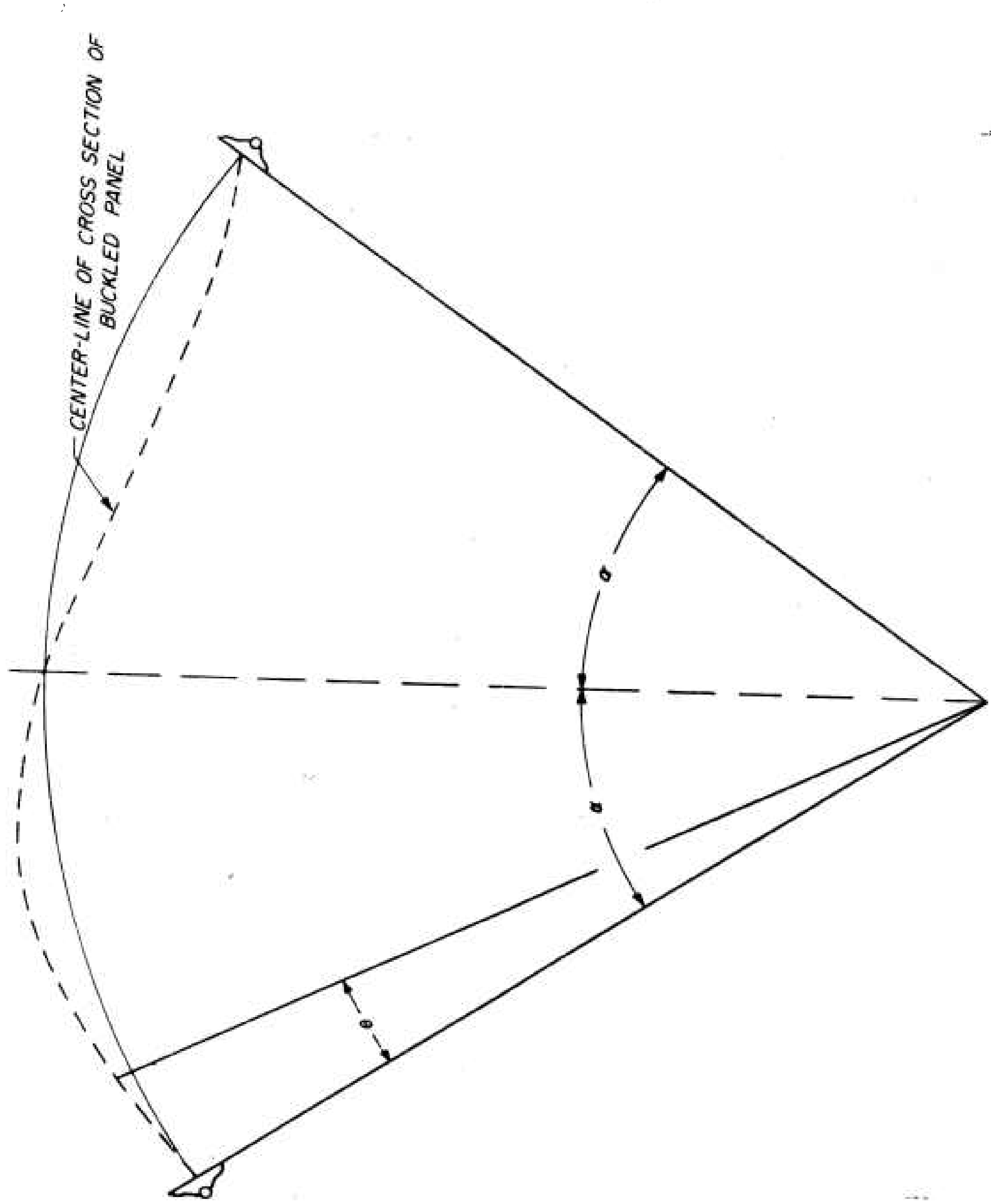

Figure 7.--Cross section of long panel.

SECTION VII

BUCKLING OF SANDWICH CYLINDERS OF FINITE LENGTH UNDER UNIFORM EXTERNAL LATERAL PRESSURE*

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Summary

A theoretical analysis is made of the problem of the buckling of circular cylinders of sandwich construction acted upon by uniform external lateral pressure. The solution obtained is based on the assumption that the sandwich cylinder is comprised of isotropic, membrane facings and an orthotropic core. The mathematical solution of the problem, which is in the form of a characteristic determinant of sixth order, is applicable to sandwich cylinders of any length and of any core thickness. Numerical results are obtained for various values of the parameters that enter the problem, and curves are included which illustrate how the critical load varies as the values of these parameters are varied.

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Introduction

Sandwich construction, as it is usually employed, consists of two relatively thin sheets of a material that has comparatively high strength and stiffness properties separated by and bonded to a relatively thick layer of a lightweight material that has comparatively low strength and stiffness properties. The two outer sheets are commonly called "facings," and the inner layer is commonly called the "core." Sandwich construction results in a composite structural element having a higher strength-weight ratio than can be obtained through the use of a single homogeneous material. For this reason, its major fields of application are in structures in which weight is a prime factor, such as aircraft and guided missiles. However, the development of improved adhesives and the improvement of methods of fabrication are making its use practical in an increasingly large number of different types of structures.

With the increased use of sandwich construction has come an increased need for reliable design data. Much work of both a theoretical and an experimental nature has been done in an effort to provide this needed information. The physical properties of many of the different materials used in sandwich construction have been determined, and analyses of a number of stress and stability problems have been performed. Progress has been somewhat retarded because of the many variables that must be taken into account in sandwich analysis and the fact that the analysis of a layered system is inherently a difficult problem. A rather complete bibliography of the part of this work that pertains particularly to shells and shell-like structures is contained in a recent U. S. Navy publication (1).

The purpose of this thesis is to obtain the solution for the critical load on circular sandwich cylinders of finite length subjected to uniform external lateral pressure. The critical load is defined as the intensity of pressure at which the cylinder buckles due to lack of stiffness. This particular problem was chosen because it represents a fundamental problem in sandwich analysis that has not previously been solved. The need for such an analysis has arisen in connection with the design of certain component parts of aircraft and guided missiles. It is believed that the method of analysis used here is somewhat more rigorous than methods that have been used to obtain solutions to other problems in sandwich analysis. For this reason, the general method of approach may be of use in obtaining solutions to other new problems as well as in obtaining better solutions to some problems that have already been solved. The solution contained in this thesis represents an extension of a previous solution of the problem of the buckling of long sandwich cylinders (2).

In this thesis it is assumed that the sandwich cylinder is comprised of very thin, isotropic facings of a relatively stiff material separated by an orthotropic core. The analysis of the facings is based on membrane theory.* The core is considered to have such a low load-carrying capacity in the tangential and longitudinal directions as compared to the facings that the normal stress in the core in these directions and the shear stress in the core on planes perpendicular to the facings and in these directions may be neglected. This core assumption

*An analysis taking into account the stiffnesses of the individual facings may be made along the lines illustrated in Ref. 3.

has been widely used in sandwich analysis and is known to represent actual sandwich construction very well. The fact that no further simplifying assumptions in regard to the core are needed, other than the assumption that it behaves as an elastic continuum, is emphasized. It is felt that some analyses of sandwich construction are not sufficiently accurate in certain ranges of physical properties and dimensions because of additional simplifying assumptions made in regard to the core. The action of the core and facings is related by the assumption that their displacements are equal at the interfaces between the core and facings. These interfaces are assumed to be at the middle surfaces of the facings.

The method used for establishing the stability criterion is similar in concept to that used by Timoshenko in the analysis of the buckling of homogeneous cylinders of finite length subjected to uniform external lateral pressure.* The assumption is made that, for pressures less than the critical, the circular cylinder remains circular and the only stresses present are a uniform circumferential stress in the facings and a uniform radial stress in the core. In discussing the buckling of the sandwich cylinder, only small deflections from this uniformly compressed form of equilibrium are considered; thus, the stresses induced in the cylinder as it goes from the circular to the slightly deformed configuration may be considered small as compared to the prebuckling stresses

*See Ref. 4, Art. 83.

Notation

r, θ, z	radial, tangential, and longitudinal coordinates, respectively
a	radius to middle surface of outer facing
b	radius to middle surface of inner facing
t	thickness of each facing
ℓ	length of cylinder
E	modulus of elasticity of facings
μ	Poisson's ratio of facings
G	modulus of rigidity of facings
E_c	modulus of elasticity of core in direction normal to facings
$G_{r\theta}$	modulus of rigidity of core in $\underline{r\theta}$ plane
G_{rz}	modulus of rigidity of core in $\underline{rz}$ plane
q	intensity of uniform external lateral loading
k	$\dfrac{1}{1 + \dfrac{b}{a} - \dfrac{Et \log \frac{b}{a}}{E_c a}}$
σ_r	small normal stress in core in radial direction
$\tau_{r\theta}, \tau_{rz}$	small transverse shear stresses in core
u_c, v_c, w_c	small radial, tangential, and longitudinal core displacements, respectively
$\epsilon_{r_c}, \epsilon_{\theta_c}, \epsilon_{z_c}$	small radial, tangential, and longitudinal core normal strains, respectively
$\gamma_{r\theta_c}, \gamma_{rz_c}, \gamma_{\theta z_c}$	small radial, tangential, and longitudinal core shearing strains, respectively
n	number of waves in circumference of buckled cylinder

λ	$\frac{\pi a}{\ell}$
$\delta_{n\theta}$	$\frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$
δ_z	$\frac{E_c}{G_{rz}}$
ϵ_θ, ϵ'_θ	small normal strains in tangential direction in outer and inner facings, respectively
ϵ_z, ϵ'_z	small normal strains in longitudinal direction in outer and inner facings, respectively
u, v, w	small radial, tangential, and longitudinal displacements of outer facing
u', v', w'	small radial, tangential, and longitudinal displacements of inner facing
N_θ, N_z, $N_{\theta z}$	normal forces and shear force per unit length of outer facing
N'_θ , N'_z , $N'_{\theta z}$	normal forces and shear force per unit length of inner facing
β	$\frac{E_c a (1 - \mu^2)}{Et}$
α	$\frac{qa (1 - \mu^2)}{Et}$
log	natural logarithm
A_n, B_n, C_n, D_n, L_n, R_n	arbitrary constants

Mathematical Analysis

In the analysis of the sandwich cylinder, cylindrical coordinates r, θ, and z are used; the dimensions of the cylinder and the positive directions of the coordinates are indicated in figure 1. The radii to the middle surfaces of the outer and inner facings are denoted by a and b, respectively, and the thickness of each facing is denoted by t. The origin of the coordinate system is placed at the middle cross-section of the cylinder whose length is denoted by ℓ. Since q is considered to be positive when it acts in the positive r-direction, buckling occurs at a negative value of q.

As mentioned previously, the cylinder at the instant before buckling is assumed to be in a state of uniform compression with the same stress condition existing in each cross-section. This assumption amounts to saying that the ends of the cylinder are not supported until after the cylinder has been initially compressed by a uniform load just less than the critical load. The expressions for the pre-buckling stresses are known,* and they are shown in figure 2. The radial stress in the core is equal to $q \frac{a}{r} k$ and the circumferential force per unit length in the outer and inner facing is $qa(1 - k)$ and qak, respectively, where q is the intensity of uniform external lateral loading in the positive r direction and

$$k = \frac{1}{1 + \frac{b}{a} - \frac{Et \log \frac{b}{a}}{E_c a}}$$

*See Ref. 2.

In discussing the equilibrium of the cylinder after it has buckled into a slightly deformed shape, it is assumed that the existing stress condition differs only slightly from the stress condition shown in figure 2. The analysis of the equilibrium of a differential element of the core of the deformed cylinder and of differential elements of the facings of the deformed cylinder is considered next.

Equilibrium of the Core

A differential element of the deformed core is shown in figure 3. The stresses, $\underline{\sigma_r}$, $\underline{\tau_{r\theta}}$, and $\underline{\tau_{rz}}$, resulting from buckling are assumed to be small as compared to the pre-buckling stress, $\underline{q \frac{a}{r} k}$. As indicated previously, the assumption is made that the core and facing materials are such that the stresses, $\underline{\sigma_\theta}$, $\underline{\sigma_z}$, and $\underline{\tau_{\theta z}}$, in the core may be neglected. The effect of the changes in geometry in the core was found to be negligible; therefore, for simplification, the equilibrium equations are written on the basis of the original geometry of the element. The summation of forces in the radial direction results in the following equilibrium equation:

$$\left(- q \frac{a}{r} k - \sigma_r\right) r \, d\theta \, dz + \left(q \frac{a}{r} k + \sigma_r + \frac{d\, q \frac{a}{r} k}{dr} dr + \frac{\partial \sigma_r}{\partial r} dr\right)$$

$$(r + dr) \, d\theta \, dz + \frac{\partial \tau_{r\theta}}{\partial\theta} d\theta \, dr \, dz + \frac{\partial \tau_{rz}}{\partial r} (r + dr) \, dr \, d\theta \, dz = 0$$

If terms containing the product of more than three differentials are neglected, the above equation may be written as

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} = 0 \qquad (1)$$

The summation of forces in the tangential direction yields

$$- \tau_{r\theta}\ r\ d\theta\ dz + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} dr\right)(r + dr)\ d\theta\ dz + \tau_{r\theta}\ dr\ d\theta\ dz = 0$$

or, neglecting differentials of higher order,

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \qquad (2)$$

Finally, the summation of forces in the longitudinal direction yields

$$- \tau_{rz}\ r\ d\theta\ dz + \left(\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r} dr\right)(r + dr)\ d\theta\ dz = 0$$

or

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0 \qquad (3)$$

If it is assumed that during buckling the generators of the cylinder deflect into a half-wave of a sine curve and the circumference subdivides into n waves, the core displacements take the following form:

$$\begin{aligned} u_c &= f_1(r) \cos n\theta \ \cos \frac{\pi z}{\ell} \\ v_c &= f_2(r) \sin n\theta \ \cos \frac{\pi z}{\ell} \\ w_c &= f_3(r) \cos n\theta \ \sin \frac{\pi z}{\ell} \end{aligned} \qquad (4)$$

where u_c, v_c, and w_c represent the small displacements of the core in the r, θ, and z directions, respectively. At the ends of the cylinder u_c and $\frac{\partial^2 u_c}{\partial z^2}$ are both zero, which represents the conditions of simply-supported edges. The displacements $\underline{u_c}$, $\underline{v_c}$, and $\underline{w_c}$ are related to the core strains by the following equations*

$$\epsilon_{r_c} = \frac{\partial u_c}{\partial r} \; ; \quad \epsilon_{\theta_c} = \frac{u_c}{r} + \frac{1}{r}\frac{\partial v_c}{\partial \theta} \; ; \quad \epsilon_{z_c} = \frac{\partial w_c}{\partial z}$$

$$\gamma_{r\theta_c} = \frac{1}{r}\frac{\partial u_c}{\partial \theta} + \frac{\partial v_c}{\partial r} - \frac{v_c}{r} \; ; \quad \gamma_{rz_c} = \frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial r} \; ; \tag{5}$$

$$\gamma_{\theta z_c} = \frac{1}{r}\frac{\partial w_c}{\partial \theta} + \frac{\partial v_c}{\partial z}$$

Also, since $\underline{\sigma_\theta}$ and $\underline{\sigma_z}$ are assumed to be zero,

$$\sigma_r = E_c \epsilon_{r_c} = E_c \frac{\partial u_c}{\partial r} \tag{6}$$

in addition to

$$\tau_{r\theta} = G_{r\theta}\gamma_{r\theta_c} = G_{r\theta}\left(\frac{1}{r}\frac{\partial u_c}{\partial \theta} + \frac{\partial v_c}{\partial r} - \frac{v_c}{r}\right) \tag{7}$$

and

$$\tau_{rz} = G_{rz}\gamma_{rz_c} = G_{rz}\left(\frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial r}\right) \tag{8}$$

If the expressions for the displacements given by equations (4) are substituted into equations (6), (7), and (8) and the resulting expressions for the stresses are required to satisfy the equilibrium equations (1),

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*See Ref. 5, p. 305.

(2), and (3), the functions of r that appear in equations (4) may be determined explicitly, thus enabling equations (4) to be written as follows:

$$u_c = [A_n \frac{a^2}{r} + B_n a + C_n a \log \frac{r}{a} + D_n r] \cos n\theta \quad \cos \frac{\pi z}{\ell} \tag{9}$$

$$v_c = [\frac{A_n}{n} (\delta_{n\theta}) \frac{a^2}{r} - n\, B_n a - n\, C_n a \, (1 + \log \frac{r}{a}) + n\, D_n r \log \frac{r}{a} + L_n r] \sin n\theta \quad \cos \frac{\pi z}{\ell} \tag{10}$$

$$w_c = [\lambda\, A_n a \log \frac{r}{a} + \lambda\, B_n r + \lambda\, C_n r \, (\log \frac{r}{a} - 1) + \lambda\, D_n a \, (\frac{1}{2} \frac{r^2}{a^2} - \frac{\delta_z}{\lambda^2} \log \frac{r}{a}) + R_n a] \cos n\theta \quad \sin \frac{\pi z}{\ell} \tag{11}$$

where $\lambda = \frac{\pi a}{\ell}$

$$\delta_{n\theta} = \frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$$

$$\delta_z = \frac{E_c}{G_{rz}}$$

and A_n, B_n, C_n, D_n, L_n, and R_n are arbitrary constants.

If equations (9), (10), and (11) are substituted into equations (6), (7), and (8), the expressions for the core stresses become

$$\sigma_r = E_c \, (- A_n \frac{a^2}{r^2} + C_n \frac{a}{r} + D_n) \cos n\theta \quad \cos \frac{\pi z}{\ell} \tag{12}$$

$$\tau_{r\theta} = - E_c \frac{A_n}{n} \frac{a^2}{r^2} \sin n\theta \cos \frac{\pi z}{\ell} \qquad (13)$$

$$\tau_{rz} = - E_c \frac{D_n}{\lambda} \frac{a}{r} \cos n\theta \sin \frac{\pi z}{\ell} \qquad (14)$$

It may be easily verified that equations (12), (13), and (14) satisfy the equilibrium equations, equations (1), (2), and (3). Furthermore, because of the manner in which they were derived, it may be stated that the right-hand sides of equations (9), (10), and (11) are unique functions insofar as their dependence upon r is concerned.

Equilibrium of the Facings

Differential elements of the facings of the deformed cylinder are shown in figure 4. The quantities N_θ, N'_θ, N_z, N'_z, $N_{\theta z}$, and $N'_{\theta z}$ represent normal and shear forces per unit length of the facing upon which they act. The core is assumed to extend to the middle surfaces of the facings, and the stresses $(\sigma_r)_{r=a}$, $(\sigma_r)_{r=b}$, $(\tau_{r\theta})_{r=a}$, $(\tau_{rz})_{r=a}$, $(\tau_{r\theta})_{r=b}$, and $(\tau_{rz})_{r=b}$ are the stresses exerted by the core on the facings as a result of buckling. All of the above quantities are assumed to be small in comparison to the pre-buckling stresses shown in figure 4 as functions of the intensity of loading, q. The bending moments and transverse shear forces in the individual facings are neglected, in accordance with membrane theory. In formulating the equilibrium equations that apply to the facings, the effects of the rotation and stretching of the facings must be taken into account. Because of the stretching of the facings, the areas of the differential elements of the outer and inner facings become $(1 + \epsilon_\theta)(1 + \epsilon_z)\, a\, d\theta\, dz$ and $(1 + \epsilon'_\theta)(1 + \epsilon'_z)\, b\, d\theta\, dz$, respectively. Since the middle surface strains are small quantities, their products may be neglected, and the differential areas of the outer and inner facings are expressed as $(1 + \epsilon_\theta + \epsilon_z)\, a\, d\theta\, dz$ and $(1 + \epsilon'_\theta + \epsilon'_z)\, b\, d\theta\, dz$, respectively. The difference in the amount of rotation of the sides AB and CD with respect to the z-axis of the cylinder causes the central angle of the outer element to change from $d\theta$ to $\left(1 + \frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{1}{a}\frac{\partial^2 u}{\partial \theta^2}\right) d\theta$. Similarly, the central angle of the inner element changes from $d\theta$ to $\left(1 + \frac{1}{b}\frac{\partial v'}{\partial \theta} - \frac{1}{b}\frac{\partial^2 u'}{\partial \theta^2}\right) d\theta$. Also, the difference in the amount of rotation of the sides AB and CD with respect to the r-axis is $\left(\frac{\partial^2 v}{\partial \theta \partial z} + \frac{\partial u}{\partial z}\right)$; the corresponding difference in rotation of the sides A'B' and C'D' of the inner element is $\left(\frac{\partial^2 v'}{\partial \theta \partial z} + \frac{\partial u'}{\partial z}\right)$. The foregoing expressions for the differences in rotation are as given by Timoshenko* and verified by Osgood and Joseph.** Since N_z and N'_z are small quantities, the relative rotation of the sides upon which they act may be neglected; the inclusion of this effect leads only to small terms of higher order.

The necessary changes in the geometry of the facings having been established, it is now possible to write three equilibrium equations for each facing. Considering first the differential element of the outer facing, the summation of forces in the direction perpendicular to the surface of the element results in the following equation:

*See Ref. 4, Art. 79.
**See Ref. 6.

$$- [qk + (\sigma_r)_{r=a}] (1 + \epsilon_\theta + \epsilon_z) a \, d\theta \, dz + q (1 + \epsilon_\theta + \epsilon_z) a \, d\theta \, dz$$

$$- [qa (1 - k) + N_\theta] (1 + \epsilon_z) (1 + \frac{1}{a} \frac{\partial v}{\partial \theta} - \frac{1}{a} \frac{\partial^2 u}{\partial \theta^2}) d\theta \, dz = 0$$

If the relationships $\underline{\epsilon_\theta = \frac{u}{a} + \frac{1}{a} \frac{\partial v}{\partial \theta}}$ and $\underline{\epsilon_z = \frac{\partial w}{\partial z}}$ are used and terms containing the products of the small stresses and displacements are neglected, the above equation of equilibrium reduces to

$$N_\theta = - a (\sigma_r)_{r=a} + q (1 - k) (u + \frac{\partial^2 u}{\partial \theta^2}) \tag{15}$$

The summation of forces in the tangential direction perpendicular to the <u>rz</u> plane yields

$$- [qa (1 - k) + N_\theta] (1 + \epsilon_z) dz + [qa (1 - k) + N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta]$$

$$(1 + \epsilon_z) dz - N_{\theta z} (1 + \epsilon_\theta) a \, d\theta + (N_{\theta z} + \frac{\partial N_{\theta z}}{\partial z} dz) (1 + \epsilon_\theta)$$

$$a \, d\theta - (\tau_{r\theta})_{r=a} (1 + \epsilon_\theta + \epsilon_z) a \, d\theta \, dz = 0$$

If terms containing products of the small stresses and strains are neglected, the above equation reduces to

$$\frac{1}{a} \frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{\theta z}}{\partial z} = (\tau_{r\theta})_{r=a} \tag{16}$$

The summation of forces in the tangential direction parallel to the <u>z</u>-axis yields

$$- N_z(1 + \epsilon_\theta)\, a\, d\theta + (N_z + \frac{\partial N_z}{\partial z})\, (1 + \epsilon_\theta)\, a\, d\theta - N_{\theta z}\, (1 + \epsilon_z)\, dz$$

$$+ (N_{\theta z} + \frac{\partial N_{\theta z}}{\partial \theta}\, d\theta)\, (1 + \epsilon_z)\, dz - (\tau_{rz})_{r=a}\, (1 + \epsilon_\theta + \epsilon_z)$$

$$a\, d\theta\, dz - qa\, (1 - k)\, (\frac{\partial^2 v}{\partial\theta\partial z} + \frac{\partial u}{\partial z})\, d\theta\, dz = 0$$

If small quantities of higher order are again neglected, the above equation of equilibrium reduces to

$$\frac{\partial N_z}{\partial z} + \frac{1}{a}\frac{\partial N_{\theta z}}{\partial \theta} = (\tau_{rz})_{r=a} + q\, (1 - k)\, (\frac{\partial^2 v}{\partial\theta\partial z} + \frac{\partial u}{\partial z}) \qquad (17)$$

The three equilibrium equations pertaining to the inner facing are obtained in a similar manner and may be written as follows:

$$N'_\theta = b\, (\sigma_r)_{r=b} + q\, \frac{a}{b}\, k\, (u' + \frac{\partial^2 u'}{\partial \theta^2}) \qquad (18)$$

$$\frac{1}{b}\frac{\partial N'_\theta}{\partial \theta} + \frac{\partial N'_{\theta z}}{\partial z} = - (\tau_{r\theta})_{r=b} \qquad (19)$$

and

$$\frac{\partial N'_z}{\partial z} + \frac{1}{b}\frac{\partial N'_{\theta z}}{\partial \theta} = -(\tau_{rz})_{r=b} + q\, \frac{a}{b}\, k\, (\frac{\partial^2 v'}{\partial\theta\partial z} + \frac{\partial u'}{\partial z}) \qquad (20)$$

On the basis of Hooke's law, the following stress-strain relations in the facings are applicable:

$$N_\theta = \frac{Et}{1-\mu^2}\, (\epsilon_\theta + \mu\epsilon_z) \qquad (21)$$

$$N_z = \frac{Et}{1-\mu^2} (\epsilon_z + \mu\epsilon_\theta) \tag{22}$$

$$N_{\theta z} = Gt\,\gamma_{\theta z} = \frac{Et}{2(1+\mu)} \gamma_{\theta z} \tag{23}$$

$$N'_\theta = \frac{Et}{1-\mu^2} (\epsilon'_\theta + \mu\epsilon'_z) \tag{24}$$

$$N'_z = \frac{Et}{1-\mu^2} (\epsilon'_z + \mu\epsilon'_\theta) \tag{25}$$

and

$$N'_{\theta z} = Gt\,\gamma'_{\theta z} = \frac{Et}{2(1+\mu)} \gamma'_{\theta z} \tag{26}$$

where $\underline{E}$, $\underline{G}$, and $\underline{\mu}$ represent the modulus of elasticity, the modulus of rigidity, and the Poisson's ratio of the facings, respectively. Since the strain-displacement relations given by equations (5) are applicable to the facings if $\underline{r}$ is replaced by $\underline{a}$ for the outer facing, $\underline{r}$ is replaced by $\underline{b}$ for the inner facing, and the corresponding displacements of the facings are substituted for the core displacements, equations (21) through (26) may be written as follows:

$$N_\theta = \frac{Et}{1-\mu^2} \left(\frac{u}{a} + \frac{1}{a}\frac{\partial v}{\partial \theta} + \mu \frac{\partial w}{\partial z}\right) \tag{27}$$

$$N_z = \frac{Et}{1-\mu^2} \left(\mu \frac{u}{a} + \frac{\mu}{a}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}\right) \tag{28}$$

$$N_{\theta z} = \frac{Et}{2(1+\mu)} \left(\frac{\partial v}{\partial z} + \frac{1}{a}\frac{\partial w}{\partial \theta}\right) \tag{29}$$

$$N_{\theta}' = \frac{Et}{1-\mu^2} \left(\frac{u'}{b} + \frac{1}{b}\frac{\partial v'}{\partial \theta} + \mu \frac{\partial w'}{\partial z}\right) \tag{30}$$

$$N_{z}' = \frac{Et}{1-\mu^2} \left(\mu \frac{u'}{b} + \frac{\mu}{b}\frac{\partial v'}{\partial \theta} + \frac{\partial w'}{\partial z}\right) \tag{31}$$

and

$$N_{\theta z}' = \frac{Et}{2(1+\mu)} \left(\frac{\partial v'}{\partial z} + \frac{1}{b}\frac{\partial w'}{\partial \theta}\right) \tag{32}$$

If continuity of displacements at the cylinder interfaces is assumed, remembering that the core is considered to extend to the middle surfaces of the facings, then

$$u = (u_c)_{r=a} \qquad u' = (u_c)_{r=b}$$

$$v = (v_c)_{r=a} \qquad v' = (v_c)_{r=b}$$

$$w = (w_c)_{r=a} \qquad w' = (w_c)_{r=b}$$

The above relations enable equations (27) through (32) to be expressed as follows:

$$N_{\theta} = \frac{Et}{1-\mu^2} \left(\frac{u_c}{a} + \frac{1}{a}\frac{\partial v_c}{\partial \theta} + \mu \frac{\partial w_c}{\partial z}\right)_{r=a} \tag{33}$$

$$N_z = \frac{Et}{1-\mu^2} \left(\mu \frac{u_c}{a} + \frac{\mu}{a} \frac{\partial v_c}{\partial \theta} + \frac{\partial w_c}{\partial z}\right)_{r=a} \tag{34}$$

$$N_{\theta z} = \frac{Et}{2(1+\mu)} \left(\frac{\partial v_c}{\partial z} + \frac{1}{a} \frac{\partial w_c}{\partial \theta}\right)_{r=a} \tag{35}$$

$$N'_\theta = \frac{Et}{1-\mu^2} \left(\frac{u_c}{b} + \frac{1}{b} \frac{\partial v_c}{\partial \theta} + \mu \frac{\partial w_c}{\partial z}\right)_{r=b} \tag{36}$$

$$N'_z = \frac{Et}{1-\mu^2} \left(\mu \frac{u_c}{b} + \frac{\mu}{b} \frac{\partial v_c}{\partial \theta} + \frac{\partial w_c}{\partial z}\right)_{r=b} \tag{37}$$

and

$$N'_{\theta z} = \frac{Et}{2(1+\mu)} \left(\frac{\partial v_c}{\partial z} + \frac{1}{b} \frac{\partial w_c}{\partial \theta}\right)_{r=b} \tag{38}$$

Also, if account is taken of the continuity of displacements at the interfaces, the equilibrium equations of the facings, equations (15) through (20), may be written as follows:

$$N_\theta = - a (\sigma_r)_{r=a} + q (1 - k) \left(u_c + \frac{\partial^2 u_c}{\partial \theta^2}\right)_{r=a} \tag{39}$$

$$\frac{1}{a} \frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{\theta z}}{\partial z} = (\tau_{r\theta})_{r=a} \tag{40}$$

$$\frac{\partial N_z}{\partial z} + \frac{1}{a} \frac{\partial N_{\theta z}}{\partial \theta} = (\tau_{rz})_{r=a} + q (1 - k) \left(\frac{\partial^2 v_c}{\partial \theta \partial z} + \frac{\partial u_c}{\partial z}\right)_{r=a} \tag{41}$$

$$N'_\theta = b\ (\sigma_r)_{r=b} + q\ \frac{a}{b}\ k\ \left(u_c + \frac{\partial^2 u_c}{\partial \theta^2}\right)_{r=b} \tag{42}$$

$$\frac{1}{b}\ \frac{\partial N'_\theta}{\partial \theta} + \frac{\partial N'_{\theta z}}{\partial z} = -\ (\tau_{r\theta})_{r=b} \tag{43}$$

and

$$\frac{\partial N'_z}{\partial z} + \frac{1}{b}\ \frac{\partial N'_{\theta z}}{\partial \theta} = -\ (\tau_{rz})_{r=b} + q\ \frac{a}{b}\ k\ \left(\frac{\partial^2 v_c}{\partial \theta \partial z} + \frac{\partial u_c}{\partial z}\right)_{r=b} \tag{44}$$

If the expressions for the facing stresses given by equations (33) through (38) are substituted into the above equations and then a further substitution is made of the expressions for the core displacements and stresses given by equations (9) through (14), six equations containing the parameters $\underline{A_n}$, $\underline{B_n}$, $\underline{C_n}$, $\underline{D_n}$, $\underline{L_n}$, and $\underline{R_n}$ that appear in the equations for the displacements of the core, equations (4), are obtained. These equations may be written as follows:

$$[1 - \frac{\beta}{n^2} + \delta_{n\theta} (1 + \frac{1-\mu}{2} \frac{\lambda^2}{n^2})] A_n + [- (n^2 - 1) + \mu\lambda^2] B_n + [- n^2 - \lambda^2] C_n + [1 + (\frac{1+\mu}{4}) \lambda^2] D_n$$

$$+ [n + (\frac{1-\mu}{2}) \frac{\lambda^2}{n}] L_n + [(\frac{1+\mu}{2}) \lambda] R_n = 0 \qquad (45)$$

$$[1 + \delta_{n\theta} - \beta + (n^2 - 1) \alpha (1 - k)] A_n + [- (n^2 - 1) + \mu\lambda^2 + (n^2 - 1) \alpha (1 - k)] B_n + [- n^2 - \mu\lambda^2$$

$$+ \beta] C_n + [1 + \frac{\mu\lambda^2}{2} + \beta + (n^2 - 1) \alpha (1 - k)] D_n + n L_n + \mu\lambda R_n = 0 \qquad (46)$$

$$[1 + \frac{\beta}{n^2} \frac{b}{a} + \delta_{n\theta} (1 + \frac{1-\mu}{2} \frac{\lambda^2}{n^2} \frac{b^2}{a^2}) + (\frac{1+\mu}{2}) \lambda^2 \frac{b^2}{a^2} \log \frac{b}{a}] A_n + [- (n^2 - 1) \frac{b}{a} + \mu\lambda^2 \frac{b^3}{a^3}] B_n + [- n^2 \frac{b}{a}$$

$$- \lambda^2 \frac{b^3}{a^3} - (n^2 - 1) \frac{b}{a} \log \frac{b}{a} + \mu\lambda^2 \frac{b^3}{a^3} \log \frac{b}{a}] C_n + [\frac{b^2}{a^2} + (\frac{1+\mu}{4}) \lambda^2 \frac{b^4}{a^4} + n^2 \frac{b^2}{a^2} \log \frac{b}{a}$$

$$+ (\frac{1-\mu}{2}) \lambda^2 \frac{b^4}{a^4} \log \frac{b}{a} - (\frac{1+\mu}{2}) \delta_z \frac{b^2}{a^2} \log \frac{b}{a}] D_n + [n \frac{b^2}{a^2} + (\frac{1-\mu}{2}) \frac{\lambda^2}{n} \frac{b^4}{a^4}] L_n + [(\frac{1+\mu}{2}) \lambda \frac{b^2}{a^2}] R_n = 0 \qquad (47)$$

$$[1 + \delta_{n\theta} + \beta \frac{b}{a} + \mu\lambda^2 \frac{b^2}{a^2} \log \frac{b}{a} + (n^2 - 1)\, \alpha k]\, A_n + [-(n^2 - 1) \frac{b}{a} + \mu\lambda^2 \frac{b^3}{a^3} + (n^2 - 1)\, \alpha k \frac{b}{a}]\, B_n$$

$$+ [-n^2 \frac{b}{a} - (n^2 - 1) \frac{b}{a} \log \frac{b}{a} - \mu\lambda^2 \frac{b^3}{a^3} + \mu\lambda^2 \frac{b^3}{a^3} \log \frac{b}{a} - \beta \frac{b^2}{a^2} + (n^2 - 1)\, \alpha k \frac{b}{a} \log \frac{b}{a}]\, C_n + [\frac{b^2}{a^2}$$

$$+ n^2 \frac{b^2}{a^2} \log \frac{b}{a} + \frac{\mu\lambda^2}{2} \frac{b^4}{a^4} - \mu\delta_z \frac{b^2}{a^2} \log \frac{b}{a} - \beta \frac{b^3}{a^3} + (n^2 - 1)\, \alpha k \frac{b^2}{a^2}]\, D_n + n \frac{b^2}{a^2}\, L_n + \mu\lambda \frac{b^2}{a^2}\, R_n = 0 \tag{48}$$

$$[\mu + (\frac{1+\mu}{2})\, \delta_{n\theta} - (\delta_{n\theta} + 1)\, \alpha\, (1 - k)]\, A_n + [-\mu\, (n^2 - 1) + \lambda^2 + (n^2 - 1)\, \alpha\, (1 - k)]\, B_n + [-n^2 - \lambda^2$$

$$+ n^2 \alpha\, (1 - k)]\, C_n + [\mu + (\frac{1-\mu}{4})\, n^2 + \frac{\lambda^2}{2} - \frac{\beta}{\lambda^2} - \alpha\, (1 - k)]\, D_n + [(\frac{1+\mu}{2})\, n - n\alpha\, (1 - k)]\, L_n$$

$$+ [\lambda + (\frac{1-\mu}{2}) \frac{n^2}{\lambda}]\, R_n = 0 \tag{49}$$

$$
\begin{aligned}
&[\mu + (\tfrac{1+\mu}{2})\,\delta_{n\theta} + (\tfrac{1-\mu}{2})\,n^2 \log \tfrac{b}{a} + \lambda^2 \tfrac{b^2}{a^2} \log \tfrac{b}{a} - (\delta_{n\theta} + 1)\,\alpha k\,]\,A_n + [-\,\mu\,(n^2 - 1)\,\tfrac{b}{a} + \lambda^2 \tfrac{b^3}{a^3} \\
&+ (n^2 - 1)\,\alpha k\,\tfrac{b}{a}]\,B_n + [-\,n^2 \tfrac{b}{a} - \lambda^2 \tfrac{b^3}{a^3} - \mu\,(n^2 - 1)\,\tfrac{b}{a} \log \tfrac{b}{a} + \lambda^2 \tfrac{b^3}{a^3} \log \tfrac{b}{a} + n^2\,\alpha k\,\tfrac{b}{a} \\
&+ (n^2 - 1)\,\alpha k\,\tfrac{b}{a} \log \tfrac{b}{a}]\,C_n + [\mu \tfrac{b^2}{a^2} + (\tfrac{1-\mu}{4})\,n^2 \tfrac{b^2}{a^2} + \tfrac{\lambda^2}{2} \tfrac{b^4}{a^4} + (\tfrac{1+\mu}{2})\,n^2 \tfrac{b^2}{a^2} \log \tfrac{b}{a} - (\tfrac{b^2}{a^2} \\
&+ \tfrac{1-\mu}{2} \tfrac{n^2}{\lambda^2})\,\delta_z \log \tfrac{b}{a} + \tfrac{\beta}{\lambda^2}\ \tfrac{b}{a} - (1 + n^2 \log \tfrac{b}{a})\,\alpha k\,\tfrac{b^2}{a^2}]\,D_n + [(\tfrac{1+\mu}{2})\,n\,\tfrac{b^2}{a^2} - n\,\alpha k\,\tfrac{b^2}{a^2}]\,L_n \\
&+ [\lambda \tfrac{b^2}{a^2} + (\tfrac{1-\mu}{2})\,\tfrac{n^2}{\lambda}]\,R_n = 0
\end{aligned}
\tag{50}
$$

where $\beta = \dfrac{E_c a\,(1-\mu^2)}{Et}$

$$\alpha = \frac{qa\,(1-\mu^2)}{Et}$$

and, as defined previously,

$$\delta_{n\theta} = \frac{E_c}{2G_{r\theta}} - \frac{n^2}{2}$$

$$\delta_z = \frac{E_c}{G_{rz}}$$

$$\lambda = \frac{\pi a}{\ell}$$

Equations (45) through (50) are satisfied if the constants A_n, B_n, C_n, D_n, L_n, and R_n are all equal to zero. This represents the uniformly compressed circular form of equilibrium of the cylinder. A buckled form of equilibrium is possible only if equations (45) through (50) yield non-zero solutions for the constants. This requires that the determinant of the coefficients of these constants be equal to zero. This determinant is shown on the following page.

$-\beta\left(\frac{n^2-1}{n^2}\right) - \left(\frac{1-\mu}{2}\right)\frac{\lambda^2}{n^2}\delta_{n\theta} + (n^2-1)\alpha(1-k)$	$(n^2-1)\alpha(1-k)$	$\beta + (1-\mu)\lambda^2$	$\beta -$
$\beta\frac{b}{a}\left(\frac{n^2-1}{n^2}\right) - \left(\frac{1-\mu}{2}\right)\frac{\lambda^2}{n^2}\frac{b^2}{a^2}\delta_{n\theta} - \left(\frac{1-\mu}{2}\right)\lambda^2\frac{b^2}{a^2}\log\frac{b}{a} + (n^2-1)\alpha k$	$(n^2-1)\alpha k\frac{b}{a}$	$-\beta\frac{b^2}{a^2} + (1-\mu)\lambda^2\frac{b^3}{a^3} + (n^2-1)\alpha k\frac{b}{a}\log\frac{b}{a}$	$-\beta\frac{b}{a}$
$-\frac{\beta}{n^2} + (\delta_{n\theta}+1) + \left(\frac{1+\mu}{2}\right)\frac{\lambda^2}{n^2}\delta_{n\theta}$	$-(n^2-1) + \mu\lambda^2$	$-n^2 - \lambda^2$	$1 +$
$\frac{\beta}{n^2}\frac{a}{b} + (\delta_{n\theta}+1)\frac{a^2}{b^2} + \left(\frac{1-\mu}{2}\right)\frac{\lambda^2}{n^2}\delta_{n\theta} + \left(\frac{1+\mu}{2}\right)\lambda^2\log\frac{b}{a}$	$-(n^2-1)\frac{a}{b} + \mu\lambda^2\frac{b}{a}$	$-n^2\frac{a}{b} - \lambda^2\frac{b}{a} - (n^2-1)\frac{a}{b}\log\frac{b}{a} + \mu\lambda^2\frac{b}{a}\log\frac{b}{a}$	$1 +$
$\lambda^2\left[\mu + \left(\frac{1+\mu}{2}\right)\delta_{n\theta} - (\delta_{n\theta}+1)\alpha(1-k)\right]$	$\lambda^2\left[-\mu(n^2-1) + \lambda^2 + (n^2-1)\alpha(1-k)\right]$	$\lambda^2\left[-n^2 - \lambda^2 + n^2\alpha(1-k)\right]$	$-\beta +$
$\lambda^2\left[\mu + \left(\frac{1+\mu}{2}\right)\delta_{n\theta} + \left(\frac{1-\mu}{2}\right)n^2\log\frac{b}{a} + \lambda^2\frac{b^2}{a^2}\log\frac{b}{a} - (\delta_{n\theta}+1)\alpha k\right]$	$\lambda^2\left[-\mu(n^2-1)\frac{b}{a} + \lambda^2\frac{b^3}{a^3} + (n^2-1)\alpha k\frac{b}{a}\right]$	$\lambda^2\left[-n^2\frac{b}{a} - \lambda^2\frac{b^3}{a^3} - \mu(n^2-1)\frac{b}{a}\log\frac{b}{a} + \lambda^2\frac{b^3}{a^3}\log\frac{b}{a}\right.$ $\left. + n^2\alpha k\frac{b}{a} + (n^2-1)\alpha k\frac{b}{a}\log\frac{b}{a}\right)$	$\beta\frac{b}{a} -$ $+$

1

$$
\left.\begin{array}{l:l:l:l:l}
 & \beta + (1-\mu)\lambda^2 & \beta - \left(\frac{1-\mu}{4}\right)\lambda^2 + (n^2-1)\,\alpha\,(1-k) & -\left(\frac{1-\mu}{2}\right)\lambda^2 & -\left(\frac{1-\mu}{2}\right) \\
\hdashline
 & -\beta\,\frac{b^2}{a^2} + (1-\mu)\,\lambda^2\,\frac{b^3}{a^3} + (n^2-1)\,\alpha\,k\,\frac{b}{a}\log\frac{b}{a} & -\beta\,\frac{b^3}{a^3} - \left(\frac{1-\mu}{4}\right)\lambda^2\,\frac{b^4}{a^4} - \left(\frac{1-\mu}{2}\right)\lambda^2\,\frac{b^4}{a^4}\log\frac{b}{a} + \left(\frac{1-\mu}{2}\right)\delta_z\,\frac{b^2}{a^2}\log\frac{b}{a} + (n^2-1)\,\alpha\,k\,\frac{b^2}{a^2} & -\left(\frac{1-\mu}{2}\right)\lambda^2\,\frac{b^4}{a^4} & -\left(\frac{1-\mu}{2}\right)\frac{b^2}{a^2} \\
\hdashline
 & -n^2 - \lambda^2 & 1 + \left(\frac{1+\mu}{4}\right)\lambda^2 & n^2 + \left(\frac{1+\mu}{2}\right)\lambda^2 & \left(\frac{1+\mu}{2}\right) \\
\hdashline
 & -n^2\,\frac{a}{b} - \lambda^2\,\frac{b}{a} - (n^2-1)\,\frac{a}{b}\log\frac{b}{a} + \mu\,\lambda^2\,\frac{b}{a}\log\frac{b}{a} & 1 + \left(\frac{1+\mu}{4}\right)\lambda^2\,\frac{b^2}{a^2} + n^2\log\frac{b}{a} + \left(\frac{1-\mu}{2}\right)\lambda^2\,\frac{b^2}{a^2}\log\frac{b}{a} - \left(\frac{1+\mu}{2}\right)\delta_z\log\frac{b}{a} & n^2 + \left(\frac{1-\mu}{2}\right)\lambda^2\,\frac{b^2}{a^2} & \left(\frac{1+\mu}{2}\right) \\
\hdashline
1)\,\alpha\,(1-k)] & \lambda^2\,[-n^2 - \lambda^2 + n^2\,\alpha\,(1-k)] & -\beta + \mu\,\lambda^2 + \left(\frac{1-\mu}{4}\right)n^2\,\lambda^2 + \frac{\lambda^4}{2} - \lambda^2\,\alpha\,(1-k) & \lambda^2\,\left[\left(\frac{1+\mu}{2}\right)n^2 - n^2\,\alpha\,(1-k)\right] & +\left(\frac{1-\mu}{2}\right)n^2 + \lambda^2 \\
\hdashline
(n^2-1)\,\alpha\,k\,\frac{b}{a}] & \begin{array}{l}\lambda^2\,\left[-n^2\,\frac{b}{a} - \lambda^2\,\frac{b^3}{a^3} - \mu\,(n^2-1)\,\frac{b}{a}\log\frac{b}{a} + \lambda^2\,\frac{b^3}{a^3}\log\frac{b}{a}\right.\\ \quad\left. + n^2\,\alpha\,k\,\frac{b}{a} + (n^2-1)\,\alpha\,k\,\frac{b}{a}\log\frac{b}{a}\right]\end{array} & \begin{array}{l}\beta\,\frac{b}{a} - \left(\frac{1-\mu}{2}\right)n^2\,\delta_z\log\frac{b}{a} + \lambda^2\,\left[\mu\,\frac{b^2}{a^2} + \left(\frac{1-\mu}{4}\right)n^2\,\frac{b^2}{a^2} + \frac{\lambda^2}{2}\,\frac{b^4}{a^4}\right.\\ \quad\left. + \left(\frac{1+\mu}{2}\right)n^2\,\frac{b^2}{a^2}\log\frac{b}{a} - \delta_z\,\frac{b^2}{a^2}\log\frac{b}{a} - \alpha\,k\,\frac{b^2}{a^2} - n^2\,\frac{b^2}{a^2}\,\alpha\,k\log\frac{b}{a}\right]\end{array} & \lambda^2\,\left[\left(\frac{1+\mu}{2}\right)n^2\,\frac{b^2}{a^2} - n^2\,\alpha\,k\,\frac{b^2}{a^2}\right] & +\left(\frac{1-\mu}{2}\right)n^2 + \lambda^2\,\frac{b^2}{a^2}
\end{array}\right| = 0 \quad (51)
$$

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The lowest, real, negative value of α for which this determinant equals zero is proportional to the critical load on the sandwich cylinder; this value of α will be referred to hereafter as α_{cr}. After this value is determined, the value of the critical load is obtained from the definition of α given previously, that is,

$$\alpha_{cr} = \frac{q_{cr}\, a\, (1-\mu^2)}{Et}$$

Numerical Computations

Since the determinant in equation (51) contains the eigenvalue, α, in four of the six rows, the expansion of this determinant would result in a fourth order algebraic equation in α. Only the lowest of the four roots of this equation, the root that corresponds to the type of buckling illustrated in figure 5, is considered. Actually, for very short cylinders the root that corresponds to a face wrinkling type of buckling may become the lowest; a satisfactory analysis of this type of buckling requires that the stiffnesses of the individual facings be taken into account. Since this analysis is based on the assumption of membrane facings, no attempt is made to determine the critical loads for this type of buckling.

A literal expansion of the determinant in equation (51) was found to be impractical. It was decided that numerical solutions obtained by the use of machine methods would be the most practical way to obtain design data.* Originally, the sixth order determinant as expressed in

*The facilities of the Numerical Analysis Laboratory of the University of Wisconsin were utilized for this purpose.

equation (51) was used as a basis for numerical solution. However, due to inherent difficulties involving the subtraction of large numbers of almost equal magnitude, it was found that sufficient accuracy could not be obtained from this approach. The sixth order determinant was then reduced in literal form to the fourth order determinant shown on the following page. A procedure was devised from which the desired eigenvalue can be determined from this determinant by trial and error. For a given set of parameters containing the dimensions and physical constants of the cylinder, the assumed value of the eigenvalue is varied until the value of the determinant becomes equal to zero. In this procedure each of the twenty-four terms in the expansion of the fourth order determinant is computed independently and their sum is accumulated subsequently. This eliminates the previous difficulties in connection with loss of accuracy.

The numerical results obtained are shown in Tables 1 through 7. The values are believed to be correct to three significant figures. More significant figures can be obtained by simply increasing the number of trials used. Curves based on the values given in the tables are shown in figures 6, 7, and 8.

$[(\delta_{n\theta} + 1) - \frac{\beta}{n^2}] [\bar{C} - (\frac{1-\mu}{1+\mu}) n^2 \bar{B}] + \bar{C} \{- (\frac{1+\mu}{1-\mu}) (\frac{n^2 - 1}{n^2}) \beta + (\frac{1+\mu}{1-\mu}) (n^2 - 1) \alpha (1 - k) - \mu \frac{\lambda^2}{n^2} \delta_{n\theta}\} + \bar{B} \lambda^2 \{\mu + (\frac{2\mu}{1+\mu}) \delta_{n\theta} - \frac{2}{1+\mu} [(\delta_{n\theta} + 1) - \frac{\beta}{n^2} + (\frac{1-\mu}{2}) \frac{\lambda^2}{n^2} \delta_{n\theta}] - (\delta_{n\theta} + 1) \alpha (1 - k)\}$	$[- (n^2 - 1) + \mu \lambda^2] [\bar{C} - (\frac{1-\mu}{1+\mu}) n^2 \bar{B}] + \bar{C} (\frac{1+\mu}{1-\mu}) (n^2 - 1) \alpha (1 - k) + \bar{B} \lambda^2 [(2 + \mu) (\frac{1-\mu}{1+\mu}) (n^2 - 1) + (\frac{1}{1+\mu}) \lambda^2 + (n^2 - 1) \alpha (1 - k)]$	$- n^2 [\bar{C}$ … $+$
$(\delta_{n\theta} + 1) (\frac{1 - \frac{b^2}{a^2}}{\frac{b^2}{a^2}}) + \beta (\frac{1 + \frac{b}{a}}{\frac{b}{a}}) + \mu \lambda^2 \log \frac{b}{a} + (n^2 - 1) \alpha k (\frac{1}{\frac{b^2}{a^2}} - \frac{b}{a} + \bar{F})$	$- (n^2 - 1) (\frac{1 - \frac{b}{a}}{\frac{b}{a}}) - \mu \lambda^2 (1 - \frac{b}{a}) + (n^2 - 1) \alpha k (\frac{1}{\frac{b}{a}} - \frac{b}{a} + \bar{F})$	$- 2 \beta$ … $-$
$\bar{A} \{(\delta_{n\theta} + 1) (\frac{1 - \frac{b^2}{a^2}}{\frac{b^2}{a^2}}) + \frac{\beta}{n^2} (\frac{1 + \frac{b}{a}}{\frac{b}{a}}) + (\frac{1+\mu}{2}) \lambda^2 \log \frac{b}{a}\} + \lambda^2 \{(\frac{2}{1+\mu}) (1 - \frac{b^2}{a^2}) [(\delta_{n\theta} + 1) - \frac{\beta}{n^2} + (\frac{1-\mu}{2}) \frac{\lambda^2}{n^2} \delta_{n\theta}] + [(\frac{1-\mu}{2}) n^2 + \lambda^2 \frac{b^2}{a^2}] \log \frac{b}{a} - (\delta_{n\theta} + 1) \alpha k (1 - \frac{b}{a} + \bar{F})\}$	$\bar{A} [- (n^2 - 1) (\frac{1 - \frac{b}{a}}{\frac{b}{a}}) - \mu \lambda^2 (1 - \frac{b}{a})] + \lambda^2 \{(\frac{2}{1+\mu}) (1 - \frac{b^2}{a^2}) [- (n^2 - 1) + \mu \lambda^2] + (1 - \frac{b}{a}) [\mu (n^2 - 1)] - \lambda^2 (1 - \frac{b^3}{a^3}) + (n^2 - 1) \alpha k \bar{F}\}$	$- \bar{A} \{- n$ … $+$ … $-$ … $+$
$\bar{B} \{(\delta_{n\theta} + 1) (\frac{1 - \frac{b^2}{a^2}}{\frac{b^2}{a^2}}) + \frac{\beta}{n^2} (\frac{1 + \frac{b}{a}}{\frac{b}{a}}) + (\frac{1+\mu}{2}) \lambda^2 \log \frac{b}{a}\} + \bar{D} \{\frac{\beta}{n^2} [1 + (\frac{1+\mu}{1-\mu}) (n^2 - 1)] - \mu \frac{\lambda^2}{n^2} \delta_{n\theta} + (\delta_{n\theta} + 1) + (\frac{1+\mu}{1-\mu}) (n^2 - 1) \alpha (1 - k)\}$	$\bar{B} \{- (n^2 - 1) (\frac{1 - \frac{b}{a}}{\frac{b}{a}}) - \mu \lambda^2 (1 - \frac{b}{a})\} + \bar{D} \{- (n^2 - 1) + \mu \lambda^2 + (\frac{1+\mu}{1-\mu}) (n^2 - 1) \alpha (1 - k)\}$	$\bar{B} \{-$

where $\bar{A} = (\frac{3+\mu}{1+\mu}) n^2 + (\frac{2}{1+\mu}) \lambda^2 - (\frac{2}{1-\mu}) (\frac{n^2 \alpha k}{1 - \frac{b^2}{a^2}}) [- \frac{b}{a} (1 - \frac{b}{a}) + \bar{F}]$; $\bar{B} = 1 - \frac{\mu \lambda^2}{n^2}$, $\bar{C} = (\frac{1-\mu}{1+\mu}) n^2 + \frac{2 (1-\mu)}{1+\mu} \lambda^2 + (\frac{1-\mu}{1+\mu}) \frac{\lambda^2}{n^2} + \alpha (1 - k) \lambda^2$, $\bar{D} = (\frac{1-\mu}{2}) \frac{\lambda^2}{n^2} (1 - \frac{b^2}{a^2})$; $\bar{F} =$

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$$
\left|\begin{array}{lll}
\begin{array}{l}[-(n^2-1)+\mu\lambda^2]\,[\bar C-(\frac{1-\mu}{1+\mu})\,n^2\,\bar B]+\bar C\,(\frac{1+\mu}{1-\mu})\,(n^2-1)\,\alpha\,(1-k)\\ \quad+\bar B\,\lambda^2[(2+\mu)\,(\frac{1-\mu}{1+\mu})\,(n^2-1)+\frac{1-\mu}{1+\mu}\,\lambda^2+(n^2-1)\,\alpha\,(1-k)]\end{array} &
\begin{array}{l}-n^2\,[\bar C-(\frac{1-\mu}{1+\mu})\,n^2\,\bar B]+\bar C[(\frac{1+\mu}{1-\mu})\,\beta+\mu\lambda^2]\\ \quad+\bar B\,\lambda^2\,[(\frac{1-\mu}{1+\mu})\,(2n^2+\lambda^2)+n^2\,\alpha\,(1-k)]\end{array} &
\begin{array}{l}[\bar C-(\frac{1-\mu}{1+\mu})\,n^2\,\bar B]+\bar C\,[(\frac{1+\mu}{1-\mu})\,\beta+(\frac{1+\mu}{1-\mu})\,(n^2-1)\,\alpha\,(1-k)]\\ \quad-\bar B\,\beta+\bar B\,\lambda^2\,[-(2+\mu)\,(\frac{1-\mu}{1+\mu})-\alpha\,(1-k)]\end{array} \\[4ex]
-(n^2-1)\,(\frac{1-\frac{b}{a}}{\frac{b}{a}})-\mu\lambda^2\,(1-\frac{b}{a})+(n^2-1)\,\alpha\,k\,(\frac{1}{\frac{b}{a}}-\frac{b}{a}+\bar F) &
\begin{array}{l}-2\beta-n^2\,(\frac{1-\frac{b}{a}}{\frac{b}{a}})+\mu\lambda^2\,(1-\frac{b}{a})-[(n^2-1)\\ \quad-\mu\lambda^2\,\frac{b^2}{a^2}-(n^2-1)\,\alpha\,k]\,\frac{1}{\frac{b}{a}}\log\frac{b}{a}\end{array} &
\begin{array}{l}-\beta\,(1+\frac{b}{a})-\frac{\mu\lambda^2}{2}\,(1-\frac{b^2}{a^2})+(n^2-\mu\,\delta_z)\log\frac{b}{a}\\ \quad+(n^2-1)\,\alpha\,k\,(1-\frac{b}{a}-\bar F)\end{array} \\[4ex]
\begin{array}{l}\bar A\,[-(n^2-1)\,(\frac{1-\frac{b}{a}}{\frac{b}{a}})-\mu\lambda^2\,(1-\frac{b}{a})]+\lambda^2\Big\{(\frac{2}{1+\mu})\,(1-\frac{b^2}{a^2})\,[-(n^2-1)\\ \quad+\mu\lambda^2]+(1-\frac{b}{a})\,[\mu\,(n^2-1)]-\lambda^2\,(1-\frac{b^3}{a^3})+(n^2-1)\,\alpha\,k\,\bar F\Big\}\end{array} &
\begin{array}{l}\bar A\,\Big\{-n^2\,(\frac{1-\frac{b}{a}}{\frac{b}{a}})+\lambda^2\,(1-\frac{b}{a})-[\frac{n^2-1}{\frac{b}{a}}-\mu\lambda^2\,\frac{b}{a}]\log\frac{b}{a}\Big\}\\ \quad+\lambda^2\Big\{(\frac{2}{1+\mu})\,(1-\frac{b^2}{a^2})\,(-n^2-\lambda^2)+n^2\,(1-\frac{b}{a})+\lambda^2\,(1\\ \quad-\frac{b^3}{a^3})-[\mu\,(n^2-1)-\lambda^2\,\frac{b^2}{a^2}]\,\frac{b}{a}\log\frac{b}{a}+n^2\,\alpha\,k\,\bar F\\ \quad+(n^2-1)\,\alpha\,k\,\frac{b}{a}\log\frac{b}{a}\Big\}\end{array} &
\begin{array}{l}\beta\,(1+\frac{b}{a})-(\frac{1-\mu}{2})\,n^2\,\delta_z\log\frac{b}{a}+\bar A\,\Big\{-(\frac{1+\mu}{4})\,\lambda^2\,(1-\frac{b^2}{a^2})\\ \quad+[n^2+(\frac{1-\mu}{2})\,\lambda^2\,\frac{b^2}{a^2}-(\frac{1+\mu}{2})\,\delta_z]\log\frac{b}{a}\Big\}+\lambda^2\Big\{(\frac{2}{1+\mu})\,(1\\ \quad-\frac{b^2}{a^2})\,[1+(\frac{1+\mu}{4})\,\lambda^2]-(1-\frac{b^2}{a^2})\,[\mu+(\frac{1-\mu}{4})\,n^2+\frac{\lambda^2}{2}\,(1\\ \quad+\frac{b^2}{a^2})]+[(\frac{1+\mu}{2})\,n^2-\delta_z]\,\frac{b^2}{a^2}\log\frac{b}{a}-\alpha\,k\,[-\frac{b}{a}\,(1-\frac{b}{a})+\bar F]\\ \quad-n^2\,\alpha\,k\,\frac{b^2}{a^2}\log\frac{b}{a}\Big\}\end{array} \\[4ex]
\begin{array}{l}\bar B\,\Big\{-(n^2-1)\,(\frac{1-\frac{b}{a}}{\frac{b}{a}})-\mu\lambda^2\,(1-\frac{b}{a})\Big\}+\bar D\,\Big\{-(n^2-1)+\mu\lambda^2\\ \quad+(\frac{1+\mu}{1-\mu})\,(n^2-1)\,\alpha\,(1-k)\Big\}\end{array} &
\begin{array}{l}\bar B\,\Big\{-n^2\,(\frac{1-\frac{b}{a}}{\frac{b}{a}})+\lambda^2\,(1-\frac{b}{a})-[\frac{n^2-1}{\frac{b}{a}}-\mu\lambda^2\,\frac{b}{a}]\log\frac{b}{a}\Big\}\\ \quad+\bar D\,\Big\{(\frac{1+\mu}{1-\mu})\,\beta-n^2+\mu\lambda^2\Big\}\end{array} &
\begin{array}{l}\bar B\,\Big\{-(\frac{1+\mu}{4})\,\lambda^2\,(1-\frac{b^2}{a^2})+[n^2+(\frac{1-\mu}{2})\,\lambda^2\,\frac{b^2}{a^2}-(\frac{1+\mu}{2})\,\delta_z]\log\frac{b}{a}\Big\}\\ \quad+\bar D\,\Big\{(\frac{1+\mu}{1-\mu})\,\beta+1+(\frac{1+\mu}{1-\mu})\,(n^2-1)\,\alpha\,(1-k)\Big\}\end{array}
\end{array}\right| = 0 \qquad (52)
$$

$(\frac{1-\mu}{1+\mu})\,n^2+\frac{2\,(1-\mu)}{1+\mu}\,\lambda^2+(\frac{1-\mu}{1+\mu})\,\frac{\lambda^4}{n^2}+2\,(1-k)\,\lambda^2$; $\bar D=(\frac{1-\mu}{2})\,\frac{\lambda^2}{n^2}\,(1-\frac{b^2}{a^2})$; $\bar F=\frac{1-\mu^2}{\beta}\log\frac{b}{a}$

2

Discussion of Results

The curves shown in figures 6, 7, and 8 are sufficient to illustrate how the value of the critical load varies when the parameters representing the dimensions and physical constants of the sandwich cylinder are varied. The ratios, $\frac{E_c}{G_{r\theta}} = 4$ and $\frac{E_c}{G_{rz}} = 10$, were chosen since they represent average values of these quantities for honeycomb cores, the type most commonly used in sandwich construction. Computations were also made with the values of these ratios interchanged since either situation is possible depending upon the orientation of the core in fabrication. The numerical results shown in Tables 1 through 4 and the curves shown in figures 6 and 7 are based on the value of $\beta = 1$. This value represents a cylinder that has a weak core; that is, a core having relatively low values of E_c, $G_{r\theta}$, and G_{rz}. The numerical results shown in Tables 5 through 7 and the curves shown in figure 8 are based on the value of $\beta = 1,000$. This value of β represents a cylinder having a stiff core. Since β depends not only upon the ratio, $\frac{E_c}{E}$, but also upon the ratio, $\frac{a}{t}$, the practical range of values of β is quite wide. Most constructions have dimensions and physical properties such that they lie in the range between $\beta = 1$ and $\beta = 1,000$. All of the computations are based on the value of .3 for the Poisson's ratio of the facings. Results were obtained for only two different ratios of the radius of the inner facing to the radius of the outer facing; that is, $\frac{b}{a} = .95$ and $\frac{b}{a} = .98$. The procedure for machine computation was set up so that any or all of the values of these parameters may be changed without any difficulty.

The results indicate that, for cylinders having relatively weak cores, it is advantageous to have the higher value of the modulus of rigidity of the core in the tangential direction rather than in the longitudinal direction. The effect of the interchange of these values is evident from a comparison of the curves shown in figures 6 and 7. This effect is negligible in the case of cylinders having strong cores as evidenced by the fact that the values given in Tables 5 and 6 are essentially the same. The fact that in all cases the overall stiffness properties of the core become relatively more important as the length of the cylinder decreases may also be noted. As the length of the cylinders is increased, the values obtained for the critical loads approach the values previously obtained on the assumption that the cylinder length is infinite. This is to be expected because, if $\underline{\lambda}$ is allowed to approach zero, the characteristic determinant for this problem reduces to the determinant obtained on the basis of the assumption of infinite length.* As the cylinder decreases in length, the value of the critical load approaches the value $\underline{G_{r\theta}\,(1 - \frac{b}{a})}$ and then falls off rapidly to zero. In any particular case, if $\underline{E_c}$ is set equal to infinity, the critical load becomes equal to $\underline{G_{r\theta}\,(1 - \frac{b}{a})}$ when the length becomes zero. This limiting value for the critical load is characteristic of stability analyses of sandwich construction in general, if $\underline{E_c}$ is assumed to be infinite. Examination of the curves shown in figures 6, 7, and 8 shows that the curves representing the cylinder having a weak core reach a maximum in the neighborhood

*See Ref. 2.

of $\frac{\ell}{a} = 1$ while those representing the cylinder having a strong core continue to rise until the cylinder becomes very short. Indications are that analyses based on the assumption of $E_c = \infty$ are sufficiently accurate in most ranges, but for relatively short cylinders having weak cores such an assumption may result in serious error.

Conclusions

The stability analysis of sandwich cylinders subjected to uniform external lateral pressure presented here is believed to be sufficiently accurate for use in design. The assumption of membrane facings limits the range of applicability of the results somewhat since the effect of the stiffnesses of the individual facings may be appreciable in some sandwich cylinders, particularly short cylinders having relatively thick facings. However, for cylinders of usual sandwich construction and with a length equal to or greater than the radius, this theory is believed to be adequate. The analysis of the problem with the stiffnesses of the facings taken into account may be performed without a great amount of additional work, and such an analysis is planned for the near future. Additional curves giving the values of the critical loads for a greater number of values of the parameters entering the problem are desirable for design purposes. Such curves may be easily prepared in the future if the time and expense involved in their preparation seem justifiable. The Forest Products Laboratory is at present planning a series of tests for the purpose of correlating the results with the theoretical results obtained here.

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Table 1. Values of Critical Pressure Expressed in Terms of α.

$$\alpha = \frac{q_{cr}\, a\, (1-\mu^2)}{Et} \qquad \beta = 1$$

$$\lambda = \frac{\pi a}{\ell} \qquad \frac{E_c}{G_{r\theta}} = 4$$

$$\frac{b}{a} = .95 \qquad \frac{E_c}{G_{rz}} = 10$$

λ	n	$-\alpha$	λ	n	$-\alpha$
0	2	0.00273	2.62	8	0.0114
0.25	"	.00292	3.15	"	.0119
.40	"	.00373	3.15	9	.0117
.50	"	.00496	3.15	10	.0116
.50	3	.00564	3.15	11	.0115
.60	"	.00588	3.15	12	.0115
.80	"	.00673	3.15	13	.0115
1.00	"	.00826	3.15	14	.0115
.80	4	.00768	3.50	13	.0115
1.00	"	.00808	3.50	14	.0115
1.20	"	.00869	3.50	15	.0115
1.40	"	.00956	4.00	15	.0114
1.60	"	.0107	4.00	16	.0114
1.40	5	.00950	4.50	16	.0113
1.60	"	.00993	4.50	17	.0112
1.80	"	.0105	4.50	18	.0112
1.80	6	.0104	4.50	19	.0112
2.00	"	.0107	4.50	20	.0112
2.25	"	.0112	5.00	20	.0110
2.25	7	.0110	6.00	21	.0106
2.62	"	.0115	10.00	23	.00930

Table 2. Values of Critical Pressure Expressed in Terms of α.

$$\alpha = \frac{q_{cr}\, a\, (1-\mu^2)}{Et} \qquad \beta = 1$$

$$\lambda = \frac{\pi a}{\ell} \qquad \frac{E_c}{G_{r\theta}} = 4$$

$$\frac{b}{a} = .98 \qquad \frac{E_c}{G_{rz}} = 10$$

λ	n	$-\alpha$	λ	n	$-\alpha$
0	2	0.000522	2.25	7	0.00403
0.0523	"	.000523	3.15	"	.00552
.25	"	.000677	2.25	8	.00401
.40	"	.00144	3.15	"	.00488
.25	3	.00121	3.15	9	.00461
.40	"	.00129	3.15	10	.00450
.60	"	.00160	3.15	11	.00447
.80	"	.00236	3.50	12	.00458
.40	4	.00188	3.50	13	.00458
.80	"	.00213	3.50	14	.00457
1.20	"	.00299	4.00	15	.00462
1.20	5	.00284	4.50	16	.00471
1.60	"	.00351	4.50	17	.00470
1.60	6	.00339	4.50	18	.00469
2.25	"	.00439	6.00	23	.00460

Table 3. Values of Critical Pressure Expressed in Terms of α.

$$\alpha = \frac{q_{cr}\, a\, (1-\mu^2)}{Et} \qquad \beta = 1$$

$$\lambda = \frac{\pi a}{\ell} \qquad \frac{E_c}{G_{r\theta}} = 10$$

$$\frac{b}{a} = .95 \qquad \frac{E_c}{G_{rz}} = 4$$

λ	n	$-\alpha$	λ	n	$-\alpha$
0	2	0.00190	1.20	5	0.00492
0.25	"	.00209	1.20	6	.00483
.40	"	.00290	1.20	7	.00483
.50	"	.00413	1.40	8	.00493
.40	3	.00327	1.40	9	.00490
.50	"	.00343	1.40	10	.00490
.60	"	.00368	1.40	11	.00488
.80	"	.00458	1.40	12	.00488
.80	4	.00426	1.60	16	.00485
1.00	"	.00472	1.60	17	.00485
1.20	"	.00543	1.60	18	.00485
1.00	5	.00462	1.80	18	.00477

Table 4. Values of Critical Pressure Expressed in Terms of α.

$$\alpha = \frac{q_{cr}\ a\ (1-\mu^2)}{Et} \qquad \beta = 1$$

$$\lambda = \frac{\pi a}{l} \qquad \frac{E_c}{G_{r\theta}} = 10$$

$$\frac{b}{a} = .98 \qquad \frac{E_c}{G_{rz}} = 4$$

λ	n	$-\alpha$	λ	n	$-\alpha$
0	2	0.000432	1.20	6	0.00177
0.25	"	.000586	1.60	"	.00212
.40	"	.00135	1.60	7	.00198
.40	3	.000948	1.60	8	.00194
.60	"	.00126	1.60	9	.00193
.80	"	.00201	1.60	10	.00193
.60	4	.00128	1.60	11	.00193
.80	"	.00146	2.00	13	.00198
1.00	"	.00179	2.25	16	.00197
1.00	5	.00163	2.25	17	.00197
1.20	"	.00184	3.15	25	.00196

Table 5. Values of Critical Pressure Expressed in Terms of α.

$$\alpha = \frac{q_{cr}\, a\, (1-\mu^2)}{Et} \qquad \beta = 1000$$

$$\lambda = \frac{\pi a}{\ell} \qquad \frac{E_c}{G_{r\theta}} = 4$$

$$\frac{b}{a} = .95 \qquad \frac{E_c}{G_{rz}} = 10$$

λ	n	$-\alpha$	λ	n	$-\alpha$
0	2	0.00394	4.50	5	0.118
0.25	"	.00420	4.50	6	.117
.40	"	.00510	5.00	"	.139
.50	"	.00640	6.00	"	.194
.60	"	.00852	6.00	7	.193
.75	"	.0137	7.00	"	.257
.60	3	.0117	7.00	8	.258
.75	"	.0126	8.00	"	.332
1.00	"	.0151	9.00	"	.427
1.20	"	.0182	8.00	9	.335
1.40	"	.0224	9.00	"	.419
1.60	"	.0279	10.00	"	.524
1.80	"	.0347	10.00	10	.517
1.80	4	.0315	11.00	"	.634
2.00	"	.0352	11.00	11	.627
2.25	"	.0406	12.00	"	.755
2.62	"	.0505	12.00	12	.748
3.15	"	.0690	13.00	"	.888
2.62	5	.0540	13.00	13	.881
3.15	"	.0667	16.00	"	1.42
4.00	"	.0953			

Table 6. Values of Critical Pressure Expressed in Terms of α.

$$\alpha = \frac{q_{cr}\ a\ (1-\mu^2)}{Et}$$

$$\beta = 1000$$

$$\lambda' = \frac{\pi a}{\ell}$$

$$\frac{E_c}{G_{r\theta}} = 4$$

$$\frac{b}{a} = .98$$

$$\frac{E_c}{G_{rz}} = 10$$

λ	n	$-\alpha$	λ	n	$-\alpha$
0	2	0.000612	4.00	6	0.0197
0.105	"	.000620	5.00	"	.0292
.25	"	.000771	5.00	7	.0265
.40	"	.00153	6.00	"	.0361
.50	"	.00272	6.00	8	.0349
.50	3	.00189	7.00	"	.0452
.60	"	.00209	7.00	9	.0448
.75	"	.00261	8.00	"	.0562
1.00	"	.00422	10.00	"	.0876
1.00	4	.00386	8.00	10	.0565
1.20	"	.00444	10.00	"	.0843
1.40	"	.00526	10.00	11	.0836
1.60	"	.00637	12.00	"	.120
2.00	"	.00953	12.00	12	.118
2.00	5	.00798	12.00	13	.118
2.62	"	.0114	15.00	15	.181
3.15	"	.0156	15.00	16	.181
3.15	6	.0140			

Table 7. Values of Critical Pressure Expressed in Terms of α.

$$\alpha = \frac{q_{cr}\, a\, (1-\mu^2)}{Et} \qquad \beta = 1000$$

$$\lambda = \frac{\pi a}{\ell} \qquad \frac{E_c}{G_{r\theta}} = 10$$

$$\frac{b}{a} = .95 \qquad \frac{E_c}{G_{rz}} = 4$$

λ	n	$-\alpha$	λ	n	$-\alpha$
0	2	0.00394	3.15	5	0.0665
0.25	"	.00420	4.00	"	.0951
.40	"	.00510	4.50	"	.118
.50	"	.00640	4.50	6	.117
.60	"	.00852	5.00	"	.138
.75	"	.0137	6.00	"	.194
.60	3	.0116	6.00	7	.193
.75	"	.0126	7.00	"	.257
1.00	"	.0151	7.00	8	.258
1.20	"	.0182	8.00	"	.332
1.40	"	.0224	9.00	"	.428
1.60	"	.0279	8.00	9	.334
1.80	"	.0347	9.00	"	.419
1.80	4	.0315	10.00	"	.526
2.00	"	.0352	10.00	10	.517
2.25	"	.0405	11.00	"	.635
2.62	"	.0504	11.00	11	.626
3.15	"	.0689	12.00	"	.757
2.62	5	.0538	12.00	12	.747

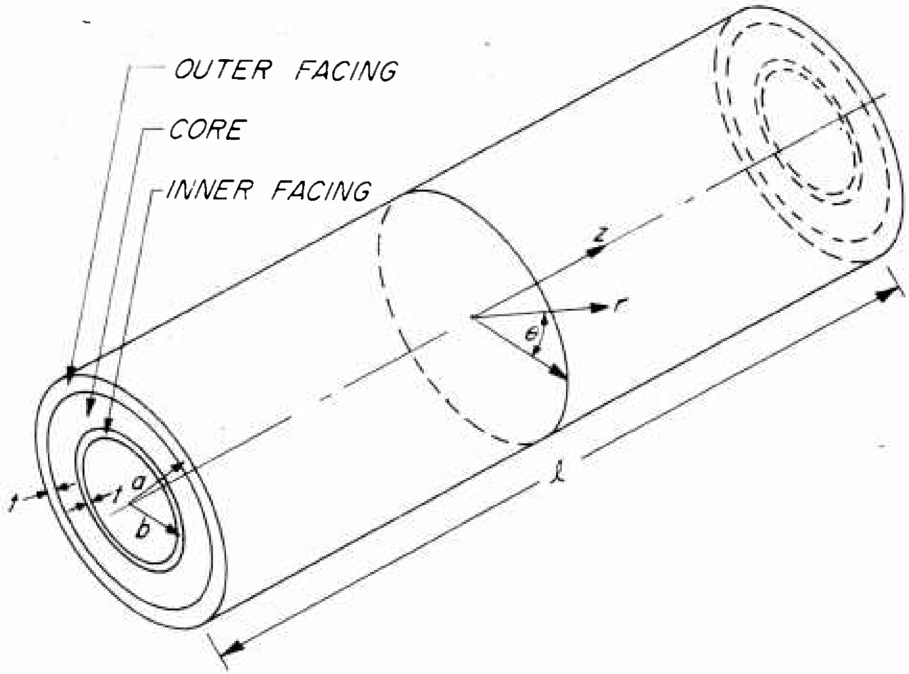

Figure 1.--Sandwich cylinder.

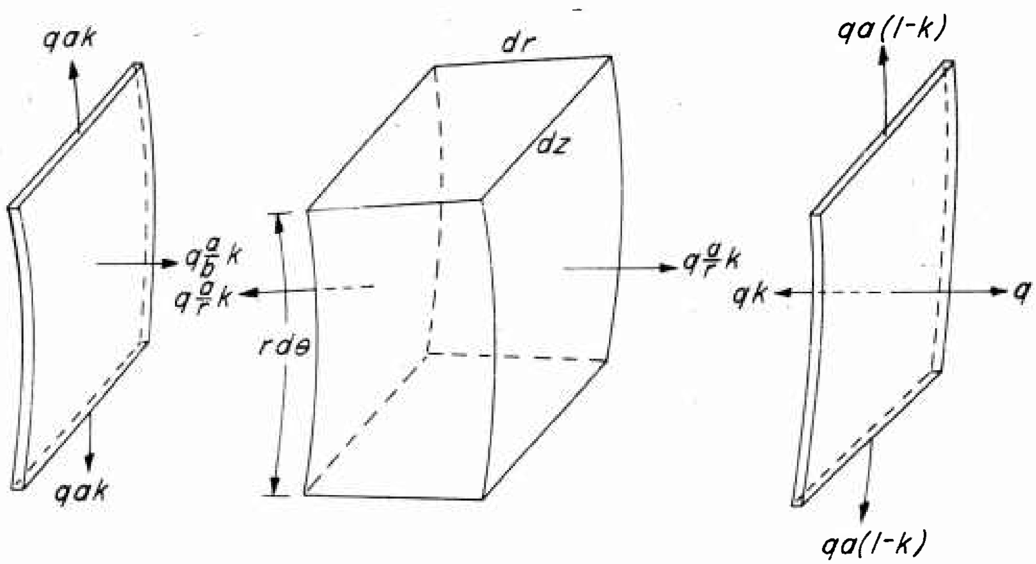

Figure 2.--Differential elements of core and facings before buckling.

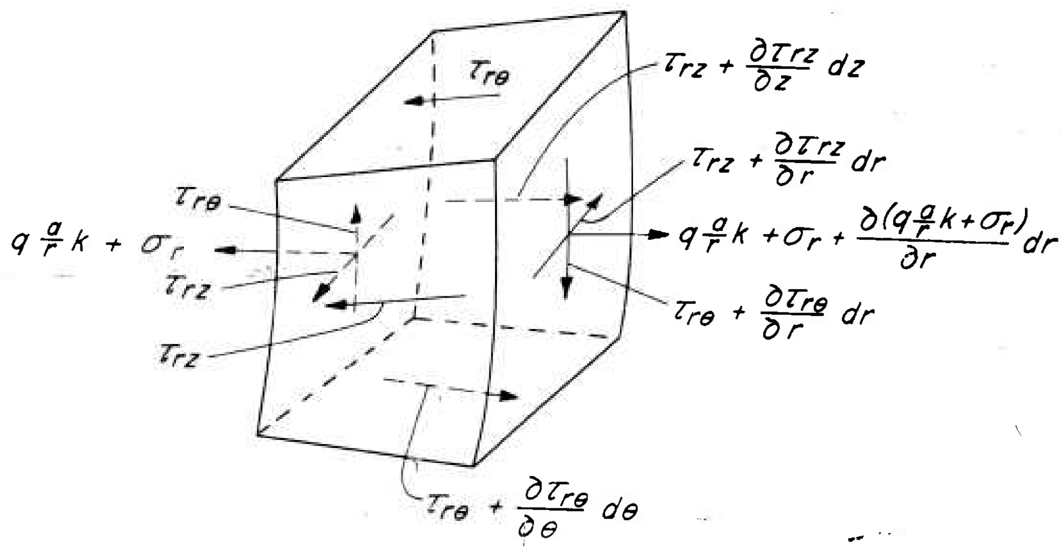

Figure 3.--Differential element of deformed core.

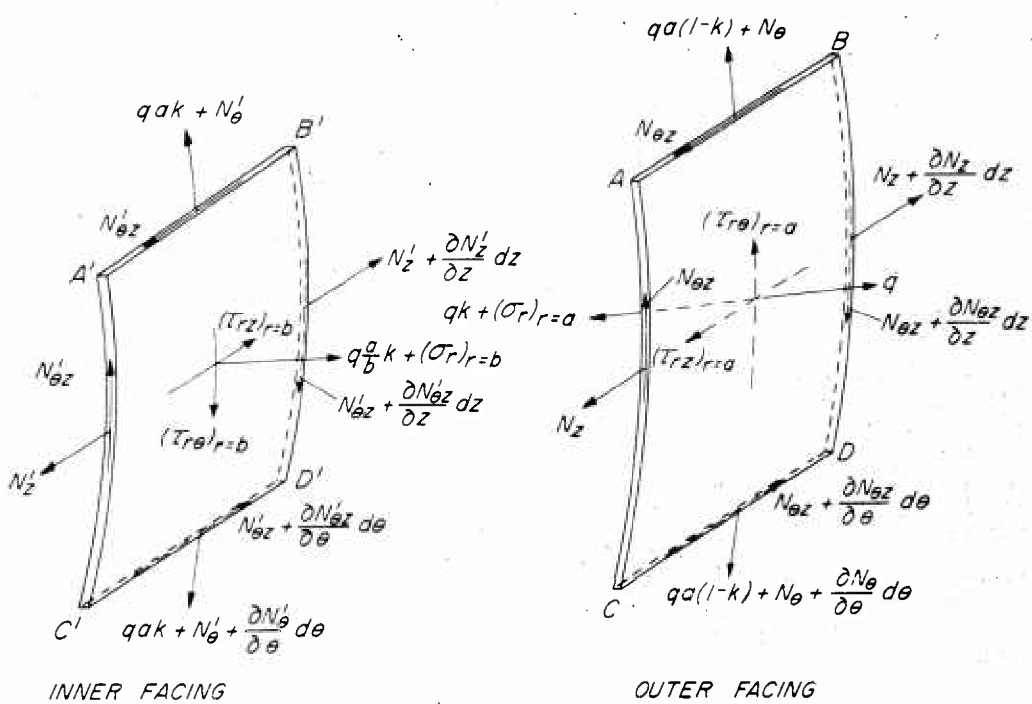

Figure 4.--Differential elements of deformed facings.

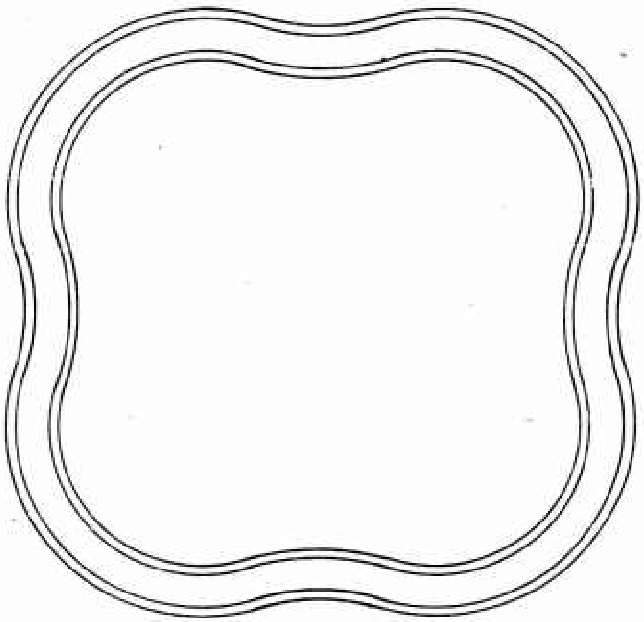

Figure 5.--Cross-section of buckled cylinder.
(for n = 4)

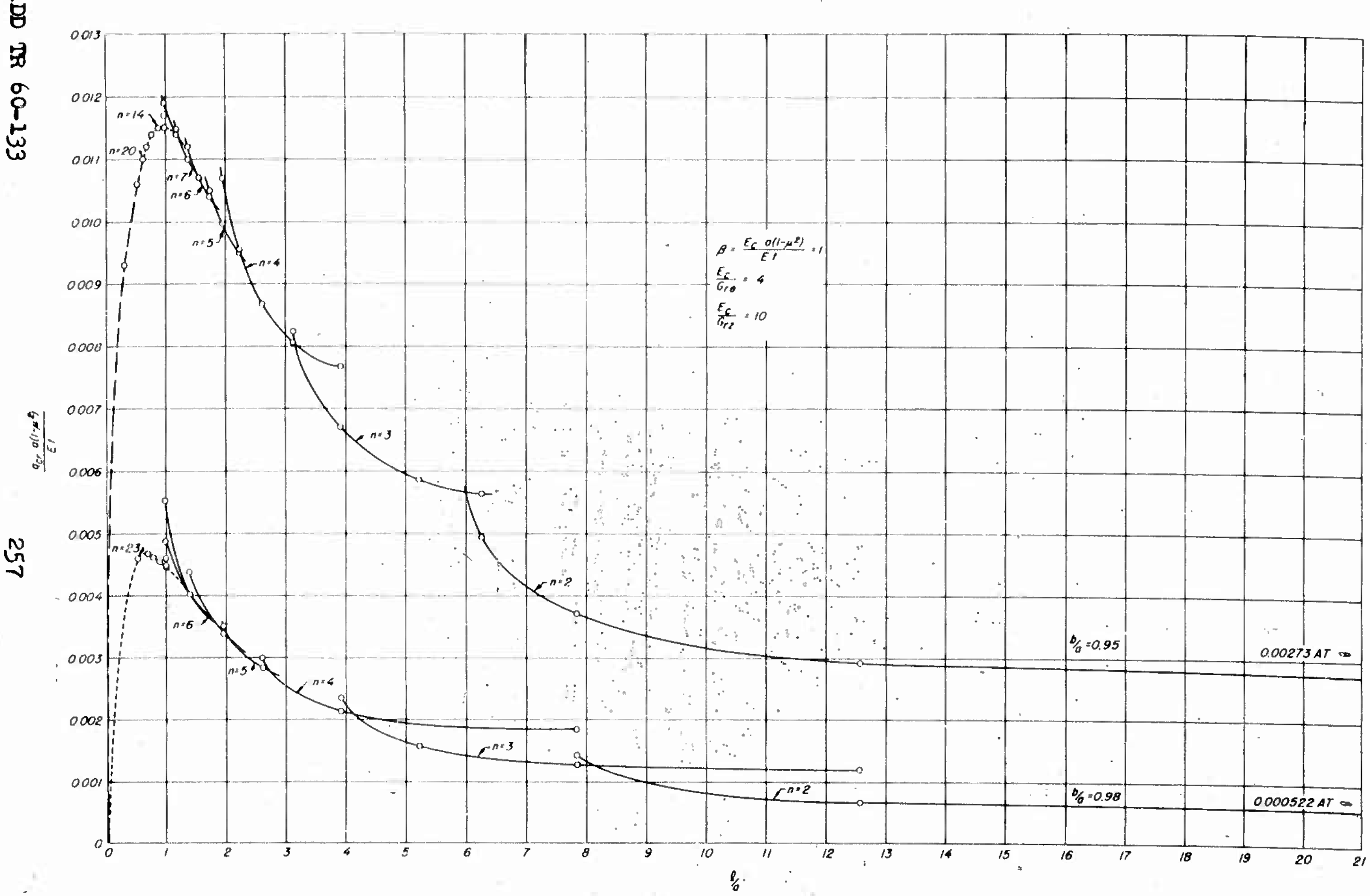

Figure 6.--Critical pressure in terms of a_{cr} versus the length to radius ratio.

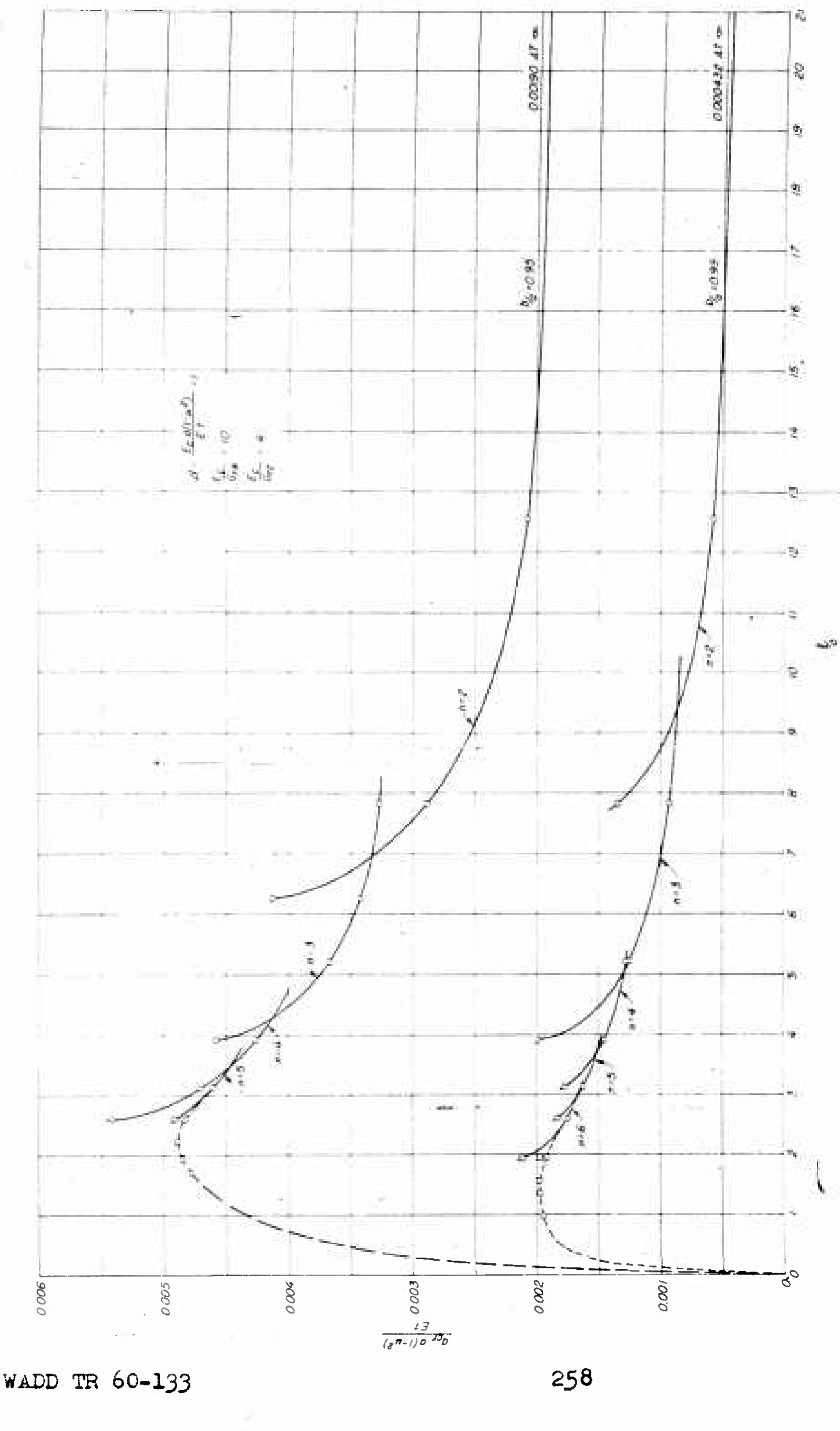

Figure 7.--Critical pressure in terms of a_{cr} versus the length to radius ratio.

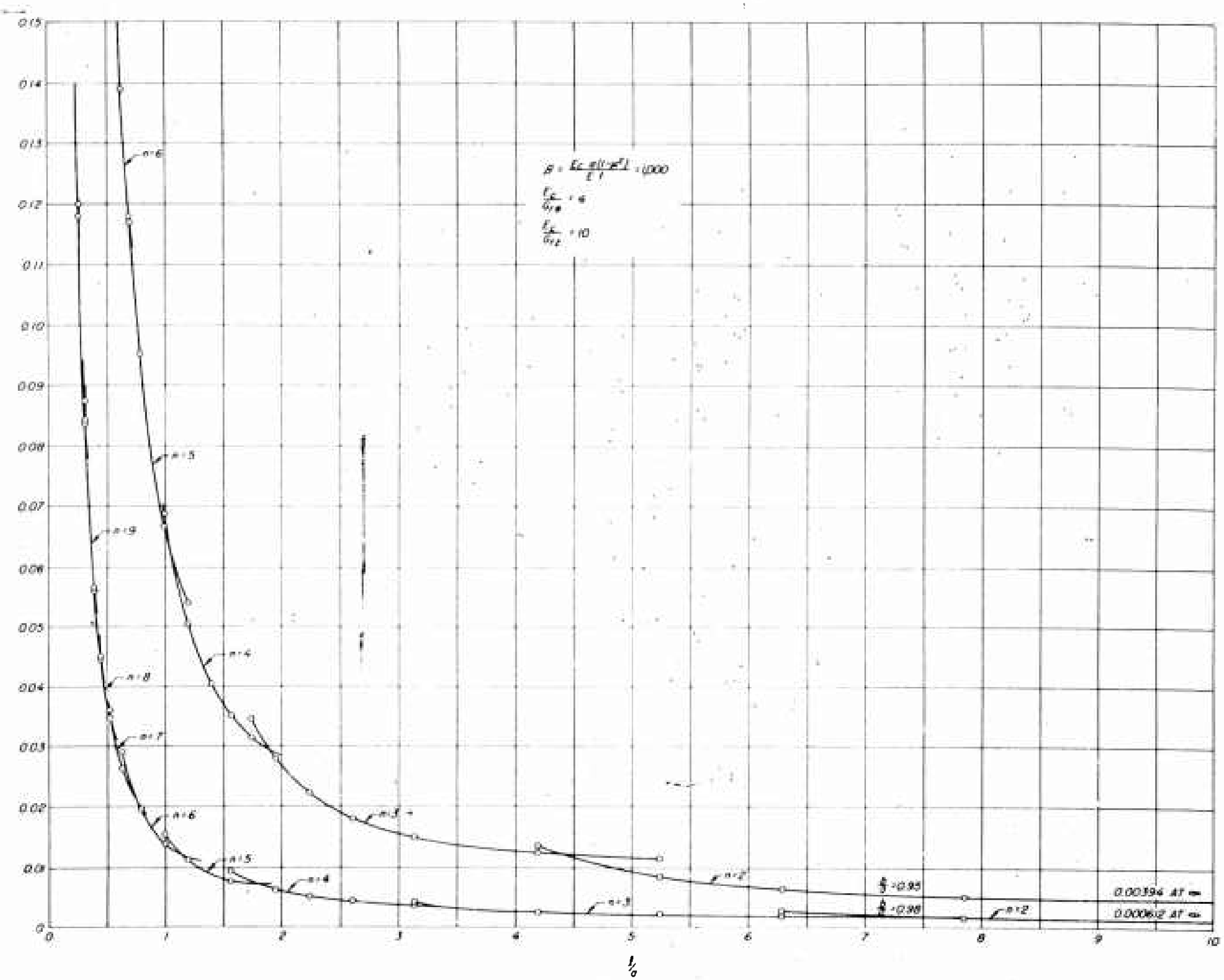

Figure 8.--Critical pressure in terms of a_{cr} versus the length to radius ratio.

SECTION VIII

DESIGN CURVES FOR THE BUCKLING OF SANDWICH CYLINDERS OF FINITE LENGTH UNDER UNIFORM EXTERNAL LATERAL PRESSURE[1]

By

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Summary

This report contains curves and formulas for the calculation of the critical external pressure of sandwich cylinders of finite length. The facings of the sandwich are equal and isotropic and their individual stiffness is not taken into account. The core is isotropic or orthotropic having natural axes in the axial, tangential, and radial directions of the cylinder. Curves are given for isotropic cores and for orthotropic cores having certain relative elastic properties. If the cores are very rigid, the method yields results that are substantially those of von Mises.

Introduction

This report presents design curves for the critical external pressure on sandwich cylinders, calculated according to the formulas developed in Forest

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[1] This progress report is one of a series (ANC-23, Item 57-3) prepared and distributed by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics Order Nos. NAer 01898 and U. S. Air Force Contract No. DO 33(616)58-1. Results here reported are preliminary and may be revised as additional data become available.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

Products Laboratory report 1844-B (9).[3] The sandwich cylinders have isotropic facings and orthotropic or isotropic cores. The natural axes of the orthotropic cores are axial, tangential, and radial. These formulas reduce substantially to those developed by von Mises (7, 10, 14) when the core is very rigid so that the stiffness of the spaced facings of the sandwich (no reduction of the stiffness due to shear strains in the core) can be used as suggested in 3.1.5 of ANC-23 (11).

A great deal of investigative work has been done on isotropic cylindrical shells subjected to external pressure since report 1844-B giving theoretical analysis of sandwich cylinders was written. It has been found that experiment sometimes yields critical loads that are less than those predicted by von Mises' theory (13). This has been attributed to two causes. First, the experimental cylinders contained imperfections that lowered the critical load (1, 2, 3, 8, 12). Second, lower energy levels are associated with post-buckling configurations of the cylinder than with those just at buckling, the former being reached without the necessity of passing through (snap-through) the latter or the energy necessary for passing through the latter being supplied by vibration or shocks (4, 5, 6, 8).

The curves published in this report do not consider snap-through buckling or cylinders with imperfections. Sandwich cylinders, however, are much more perfect than their solid counterparts because they are thicker and the effect of an imperfection is in proportion to the ratio of its amplitude to the thickness of the cylindrical shell. Also, the curves neglect the stiffnesses of the individual facings. These stiffnesses add a considerable amount to the critical loads when the cylinders are short, and it is for short cylinders that "snap-through" is likely to occur (6). Thus it seems that these curves are useful in the design of sandwich cylinders.

Development of Formula from which Design Curves Were Calculated

The critical external pressure is found by placing the determinant on page 23 of report No. 1844-B (9) equal to zero and solving for α. This determinant can be simplified if the transverse modulus of elasticity of the core (E_c) is assumed to be infinite. For most core materials except possibly for low density foams E_c is sufficiently large so that this assumption yields values of the critical pressure that are only very slightly too great.

[3] Underlined numbers in parentheses refer to the references.

Before E_c is allowed to approach infinity, the first and fourth columns of the determinant are multiplied by $\frac{G_{Rz}}{E_c}$ and the third column is divided by E_c. Then when E_c approaches infinity, the expressions in rows 3, 4, 5, and 6 in column 3 approach zero.

The expressions in each row, excepting the first, are replaced by new expressions, as indicated by the following formulas in which R represents the expression in the row designated by its subscript and in some column. The primed values are the new ones to be substituted for the old. These substitutions can be made without changing the value of α because of the well-known properties of determinants and because the determinant is equated to zero.

$$R_2' = R_2 \frac{a^2}{b^2} + R_1$$

$$R_3' = R_3 + R_2'$$

$$R_4' = R_4 + R_2'$$

$$R_5' = 2R_5 + (n^2 + 3\lambda^2)\, R_2'$$

$$R_6' = 2R_6 + (n^2 + 3\frac{b^2}{a^2}\lambda^2)\, R_2'$$

These substitutions cause the expressions in column 3 and rows 2, 3, 4, 5, and 6 and those in column 6 and rows 3, 4, 5, and 6 to become zero, and the determinant is readily reduced (by minors) from a 6-by-6 to a 4-by-4 determinant. This determinant is simplified slightly by replacing the second row of expressions by the second row minus the first row.

After the determinant was written in this form, a change in parameters was made using the following nomenclature:

R -- mean radius of the sandwich cylinder

h -- thickness of the sandwich

c -- thickness of the core of the sandwich

n -- number of half waves in the circumference of the cylinder

$$r = \frac{G_{Rz}}{G_{R\theta}}$$

$$V = \frac{E}{(1 - \mu^2)\, G_{R\theta}} \frac{t}{h}$$

$$K = \alpha \frac{h}{a} = \frac{q(1 - \mu^2)}{E} \frac{h}{t}$$

where G_{Rz} and $G_{R\theta}$ are the moduli of rigidity of the core associated with the radial and axial directions and with the radial and tangential directions; E, μ, and t are the modulus of elasticity, Poisson's ratio, and thickness of the facings; and q is the external critical pressure on the cylinder.

The radii a and b were eliminated by the following equations obtained from the geometry of the cylinder:

$$a = R + \frac{1}{4}(h + c)$$

$$b = R - \frac{1}{4}(h + c)$$

and the following substitutions made:

$$\Phi = \frac{4R}{h}$$

$$\phi = \frac{h + c}{h}$$

The expressions in the final determinant are:

Row 1, column 1

$$\frac{n^2 - 1}{n^2} \frac{\Phi + \phi}{\Phi - \phi} - 1 + \frac{2V}{\Phi + \phi} \left[1 - \frac{\pi^2}{3n^2} \left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2 \right]$$

Row 2, column 1

$$\frac{1}{n^2} \frac{2\Phi}{\Phi - \phi} + \frac{8\Phi\phi}{\Phi^2 - \phi^2} V$$

Row 3, column 1

$$\frac{n^2 - 1}{n^2} \frac{2\phi}{\Phi - \phi} \left[n^2 + 3\pi^2 \left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2 \right] + 4\pi^2 V \frac{\Phi + \phi}{\Phi^2} \left(\frac{R}{\ell}\right)^2 \left[\frac{1}{3} - \frac{\pi^2}{n^2} \left(\frac{R}{\ell}\right)^2 \left(\frac{\Phi + \phi}{\Phi}\right)^2 - \frac{\Phi^2 - \phi^2}{8\Phi} K \right]$$

Row 4, column 1

$$\frac{n^2-1}{n^2}\,\frac{2\phi}{\Phi-\phi}\left[n^2+3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi-\phi}{\Phi}\right)^2\right]+4\pi^2 V\,\frac{\Phi+\phi}{\Phi^2}\left(\frac{R}{\ell}\right)^2\left[\frac{1}{3}\right.$$

$$\left.-\frac{\pi^2}{n^2}\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi-\phi}{\Phi}\right)^2-\frac{(\Phi+\phi)^2}{8\Phi}\,K\right]$$

Row 1, column 2

$$(n^2-1)\,\frac{(\Phi+\phi)^2}{4\Phi}\,K\,\frac{\Phi^2+\phi^2}{\Phi^2-\phi^2}-n^2+1+\frac{\pi^2}{3}\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi+\phi}{\Phi}\right)^2$$

Row 2, column 2

$$-\frac{2\phi}{\Phi+\phi}\left[(n^2-1)\,\frac{\Phi+\phi}{\Phi-\phi}+\frac{\pi^2}{3}\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi+\phi}{\Phi}\right)^2\right]$$

Row 3, column 2

$$2\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi+\phi}{\Phi}\right)^2\left[-\frac{n^2-1}{3}+\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi+\phi}{\Phi}\right)^2+(n^2-1)\,\frac{\Phi^2-\phi^2}{8\Phi}\,K\right]$$

$$+(n^2-1)\,\frac{(\Phi+\phi)^2}{4\Phi}\,\frac{\Phi^2+\phi^2}{\Phi^2-\phi^2}\,K\left[n^2+3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi+\phi}{\Phi}\right)^2\right]$$

Row 4, column 2

$$2\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi+\phi}{\Phi}\right)^2\left[-\frac{n^2-1}{3}\,\frac{\Phi-\phi}{\Phi+\phi}+\pi^2\left(\frac{R}{\ell}\right)^2\frac{(\Phi-\phi)^3}{\Phi(\Phi+\phi)}+(n^2-1)\,\frac{\Phi^2-\phi^2}{8\Phi}\,K\right]$$

$$+(n^2-1)\,\frac{(\Phi+\phi)^2}{4\Phi}\,\frac{\Phi^2+\phi^2}{\Phi^2-\phi^2}\,K\left[n^2+3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi-\phi}{\Phi}\right)^2\right]$$

Row 1, column 3

$$\frac{2\phi}{\Phi+\phi}+\frac{4}{3r}\,\frac{V}{\Phi+\phi}\log\frac{\Phi-\phi}{\Phi+\phi}$$

Row 2, column 3

$$-\frac{8}{3r}\,\frac{V}{\Phi+\phi}\log\frac{\Phi-\phi}{\Phi+\phi}$$

Row 3, column 3

$$-2 + \frac{2\phi}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2\right] + \frac{4}{3r}\,\frac{V}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2\right]\log\frac{\Phi - \phi}{\Phi + \phi}$$

Row 4, column 3

$$2\,\frac{\Phi - \phi}{\Phi + \phi} + \frac{2\phi}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2\right] - \frac{4}{3r}\,\frac{V}{\Phi + \phi}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2\right]\log\frac{\Phi - \phi}{\Phi + \phi}$$

Row 1, column 4

$$n^2 - \frac{\pi^2}{3}\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2$$

Row 2, column 4

$$-\frac{4\pi^2}{3}\left(\frac{R}{\ell}\right)^2\frac{\phi}{\Phi}$$

Row 3, column 4

$$4\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2 n^2\left[\frac{1}{3} - \frac{\Phi^2 - \phi^2}{16\Phi}K\right] - \frac{2\pi^2}{3}\left(\frac{R}{\ell}\right)^2\frac{\Phi^2 + \phi^2}{\Phi^2}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi + \phi}{\Phi}\right)^2\right]$$

Row 4, column 4

$$4\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2 n^2\left[\frac{1}{3} - \frac{(\Phi + \phi)^2}{16\Phi}K\right] - \frac{2\pi^2}{3}\left(\frac{R}{\ell}\right)^2\frac{\Phi^2 + \phi^2}{\Phi^2}\left[n^2 + 3\pi^2\left(\frac{R}{\ell}\right)^2\left(\frac{\Phi - \phi}{\Phi}\right)^2\right]$$

This determinant was placed equal to zero and values of K determined by an electronic computer for various values of $\frac{\ell}{R}$, n, V, $\frac{R}{h}$, and $\frac{c}{h}$.

The limits of these curves for very long and very short cylinders may be found from Forest Products Laboratory Reports 1844-A and 1844-B (9). For infinitely long cylinders having membrane facings and for which the modulus of elasticity of the core in the radial direction is infinite, equation (72) of Report 1844-A becomes:

$$K = \frac{3\,(\Phi - \phi)\,\phi}{2\,(\Phi^2 + \phi^2)\left[\dfrac{\Phi^2 - \phi^2}{16\phi} + \dfrac{Et}{G_{R\theta}\,(1 - \mu^2)\,h}\right]}$$

For very short cylinders having membrane facings and for which the modulus of elasticity of the core in the radial direction is infinite, the equation on page 28 of report 1844-B becomes:

$$K = \frac{2\Phi}{\Phi + \phi}\;\frac{G_{R\theta}\,(1 - \mu^2)\,h}{Et}$$

From this equation the critical hoop compression per unit length of cylinder is found to be:

$$N_\theta = \frac{1}{2}\,(h + c)\,G_{R\theta}$$

which is the usual limit imposed on the edge compression of sandwich constructions with membrane facings by the shear instability of the core.

Description of Design Curves

The design curves (figs. 1 to 6) apply to sandwich cylinders having equal isotropic facings and isotropic cores or orthotropic cores having thin natural axes parallel to the axial, tangential, and radial directions of the cylinders. They are plots of K against $\frac{\ell}{R}$ where the critical pressure is given by

$$q = \frac{Et}{(1 - \mu^2)\,h}\,K$$

and E, μ, and t are the modulus of elasticity, Poisson's ratio, and thickness of the facings; h the thickness of the sandwich; ℓ and R the length and mean radius of the cylinder. Six curve sheets are presented, two for each of three values (10, 50, and 100) of $\frac{R}{h}$. One of each pair of curve sheets applies to sandwich having very thin facings for which the ratio of the thickness of the core to the thickness of the sandwich (c/h) is substantially unity. The other applies to sandwich for which this ratio is 0.7. Each curve applies to sandwich having a particular value of:

$$V = \frac{E}{(1 - \mu^2)\, G_{R\theta}} \frac{t}{h}$$

where $G_{R\theta}$ is the modulus of rigidity of the core associated with the radial and tangential directions of the cylinder. It was found that the modulus of rigidity of the core associated with the radial and axial directions (G_{Rz}) has very little influence on the critical pressure. It does not enter the formulas for the critical pressure of very long or very short cylinders. Calculations were made for cylinders having three values of

$$r = \frac{G_{Rz}}{R_{R\theta}}$$

These values were 2.5, 1.0, and 0.4 to agree with the values appropriate for some honeycomb and isotropic core materials. Many of the curves are not affected by the use of these values. Some of them are affected slightly. This is shown in the figures by the use of three adjacent curves; the greatest, intermediate, and least values of K are associated with the greatest, intermediate, and least values of r.

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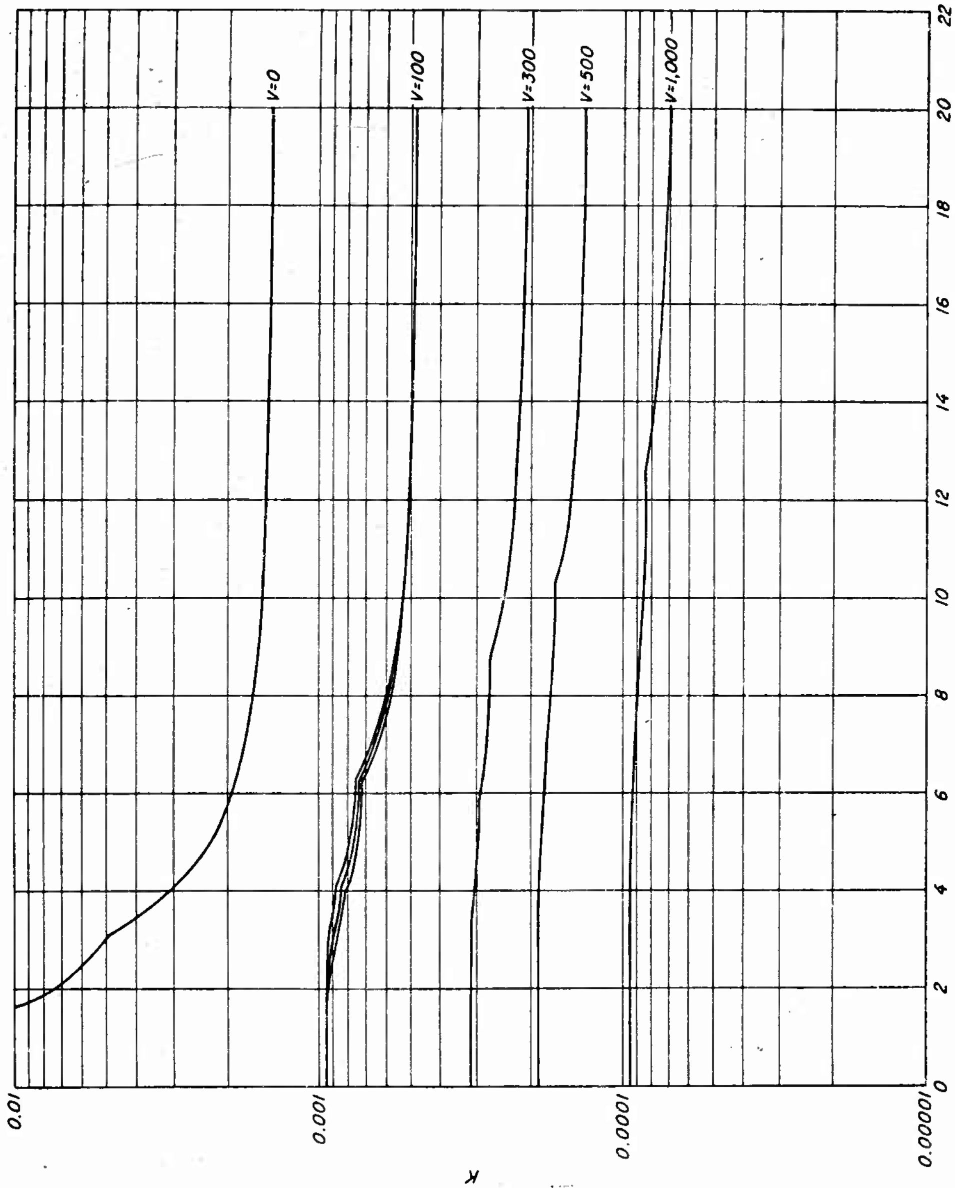


Figure 1. --Values of K for $\frac{R}{h} = 10$ and $\frac{c}{h} = 1$ for various values of V and $\frac{\ell}{R}$.

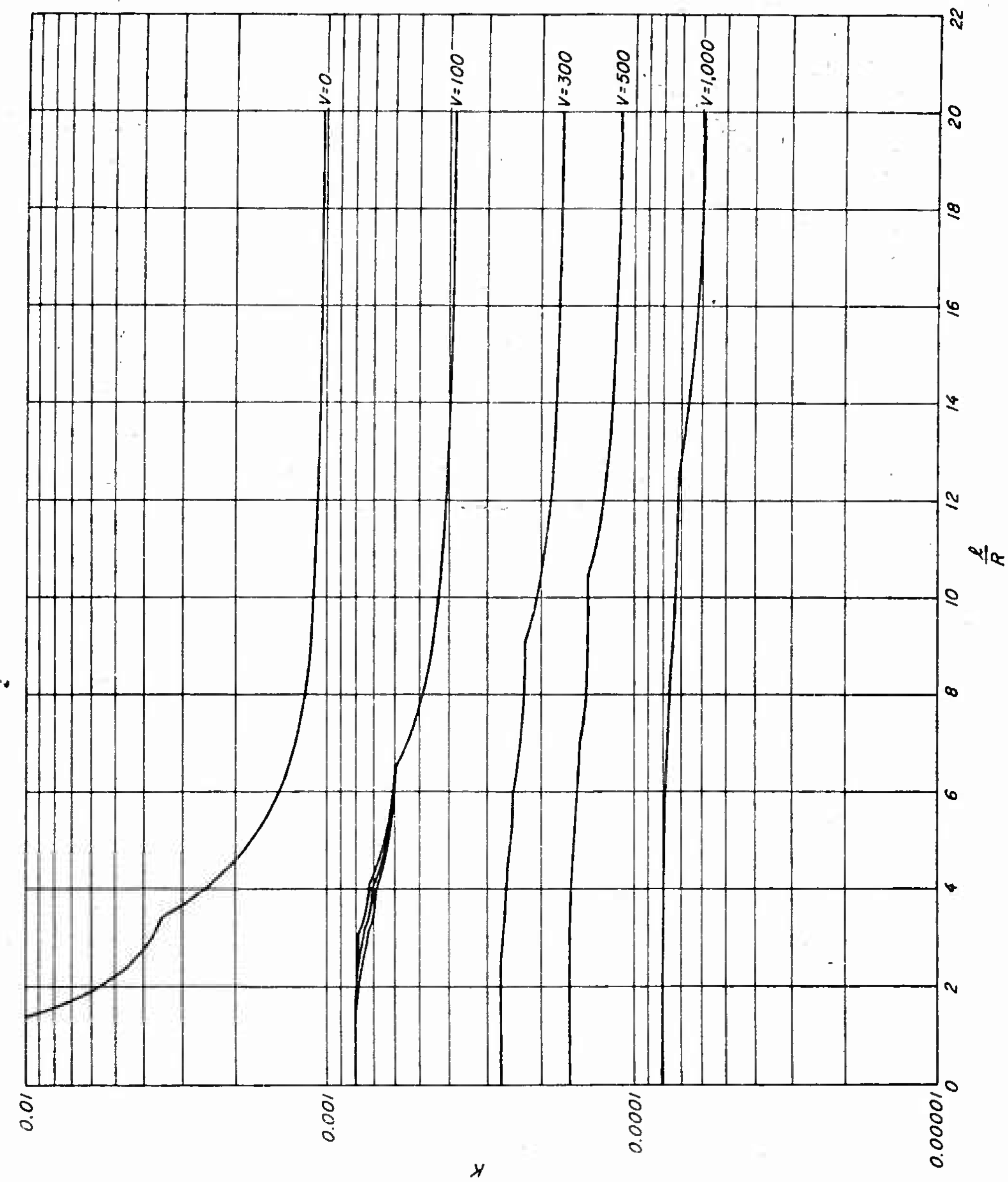


Figure 2. --Values of K for $\frac{R}{h} = 10$ and $\frac{c}{h} = 0.7$ for various values of V and $\frac{\ell}{R}$

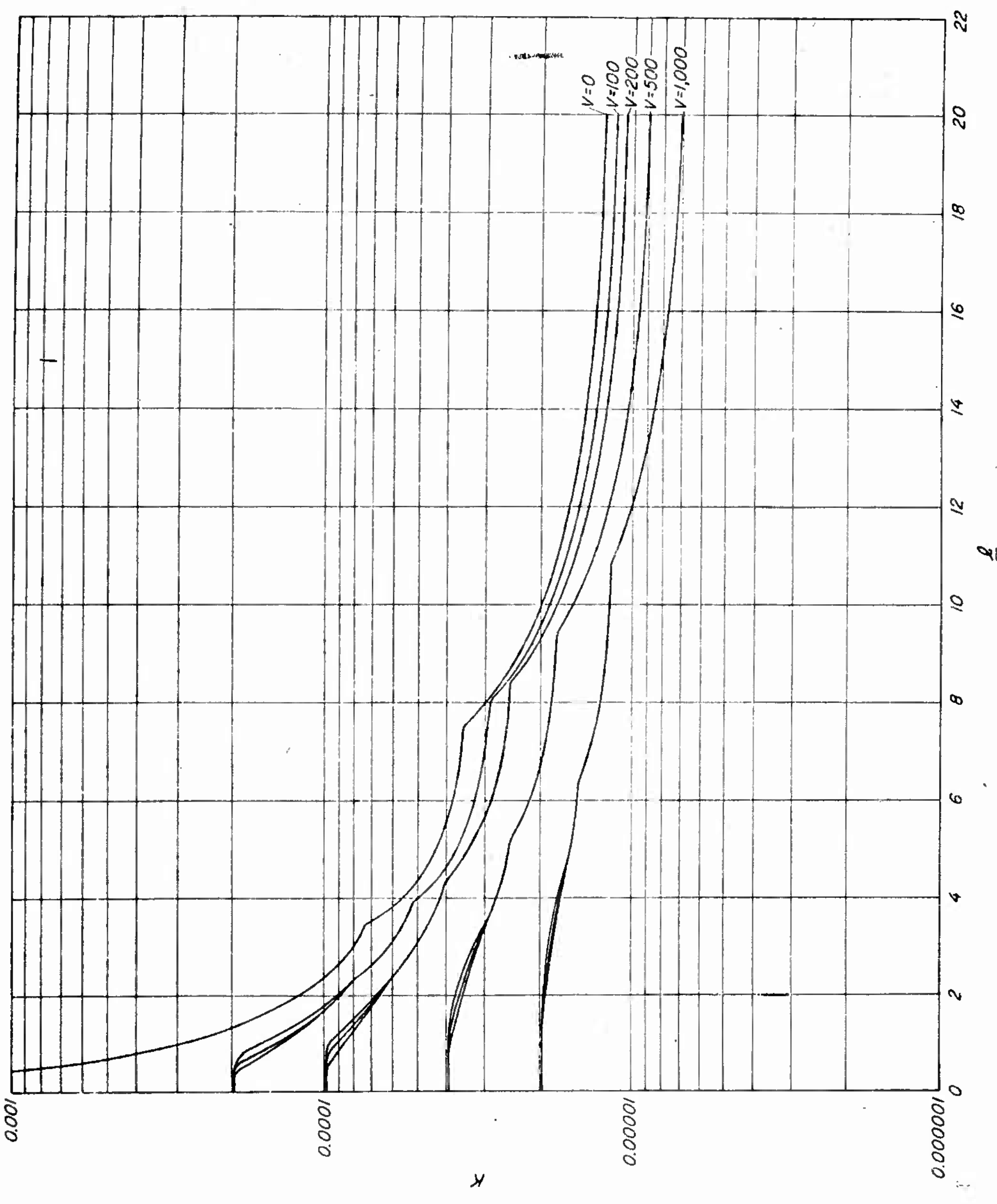


Figure 3. --Values of K for $\frac{R}{h}$ = 50 and $\frac{c}{h}$ = 1 for various values of V and $\frac{\ell}{R}$.

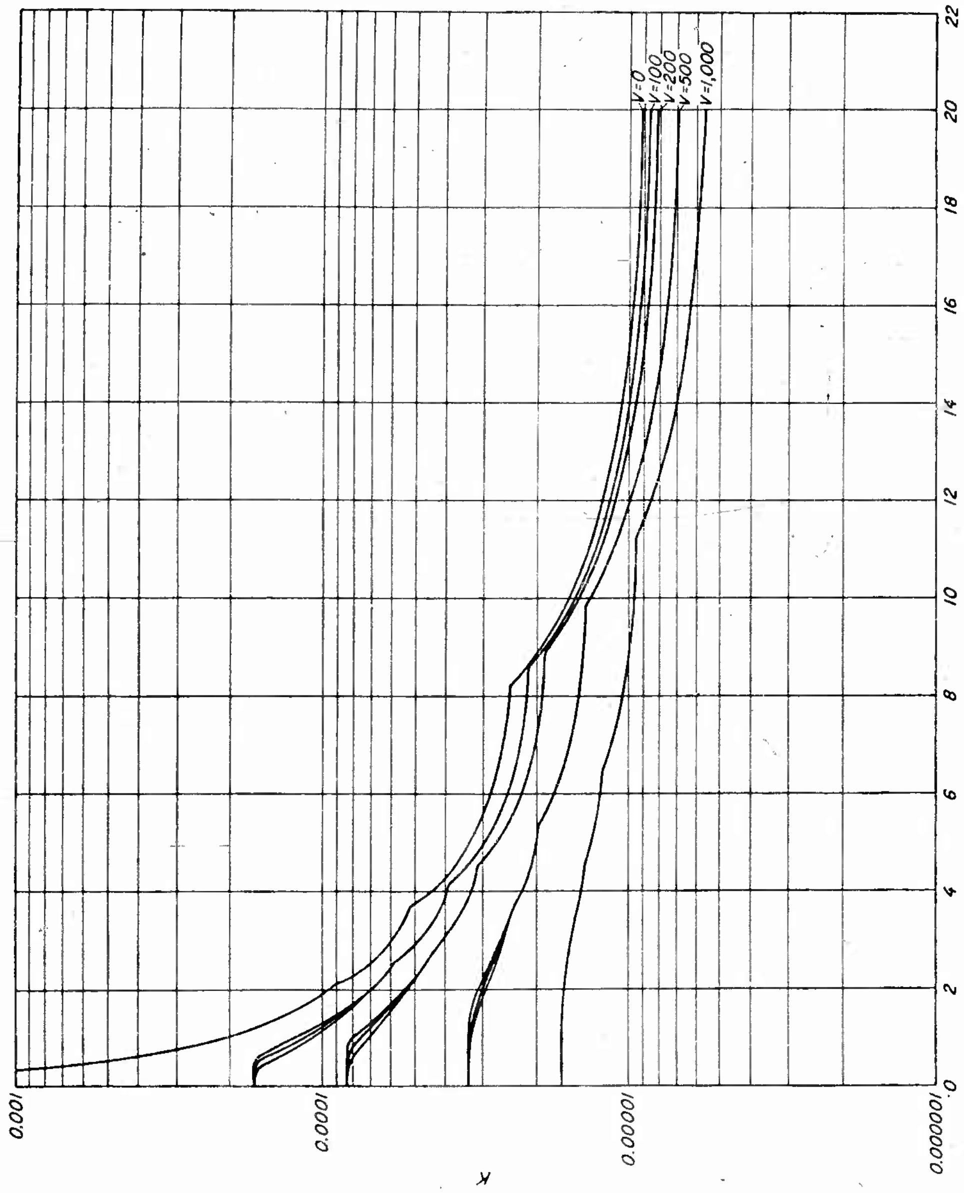
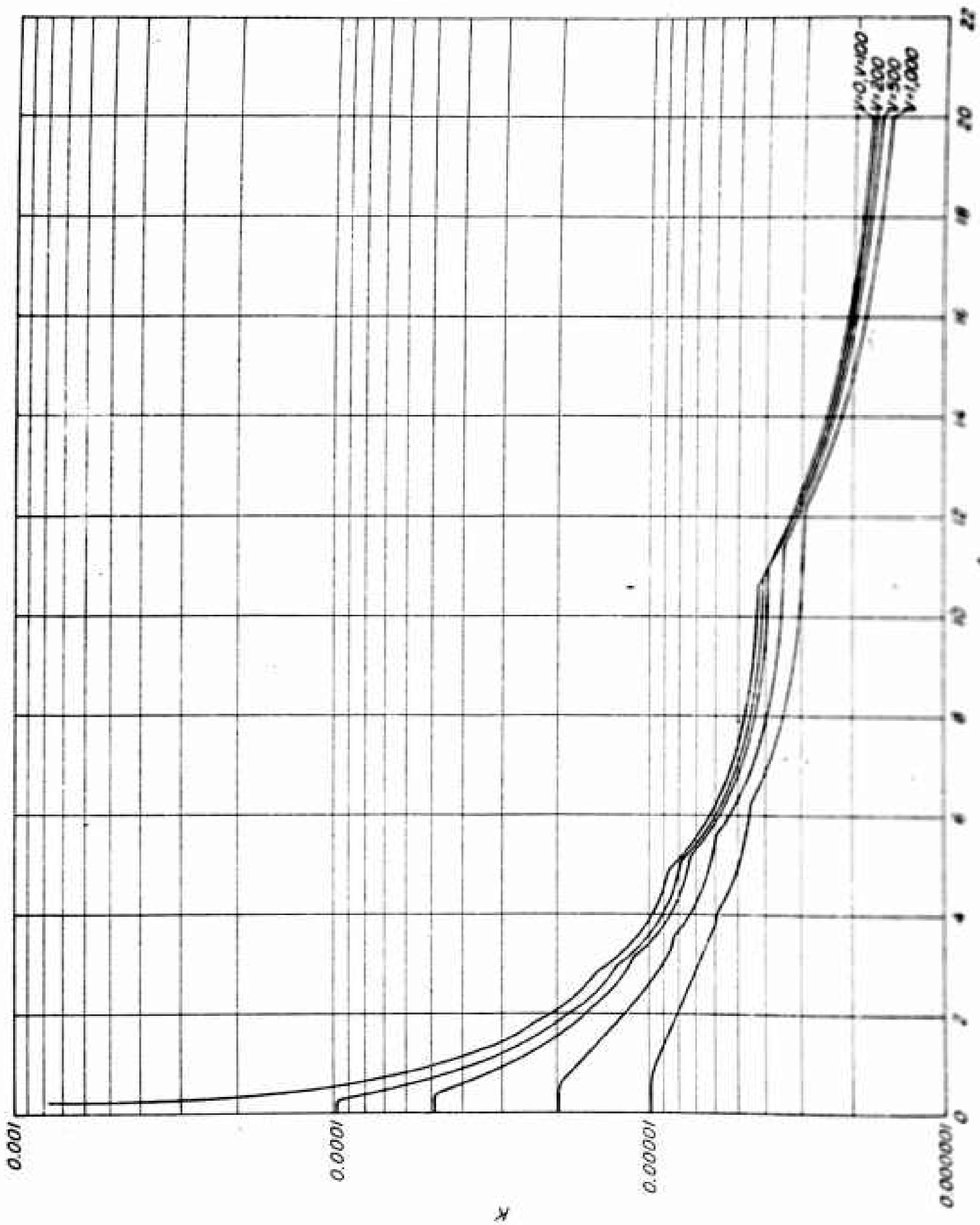


Figure 4. --Values of K for $\frac{R}{h} = 50$ and $\frac{c}{h} = 0.7$ for various values of V and $\frac{\ell}{R}$.


0.001
0.0001
0.00001
0.000001
K
0
2
4
6
8
10
12
14
16
18
20
22
V=0,V=100
V=200
V=500
V=1,000

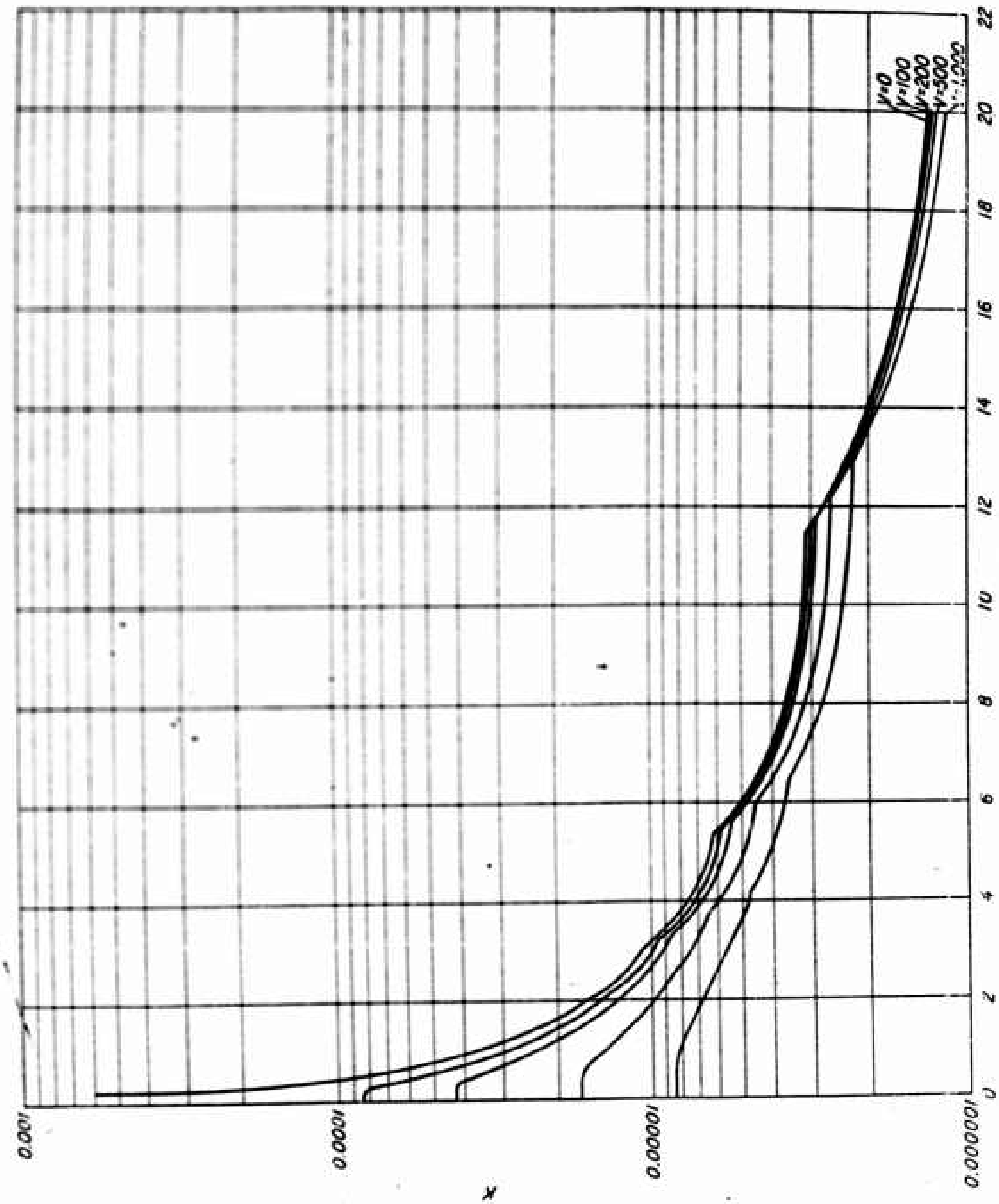


Figure 6. --Values of K for $\frac{R}{h} = 100$ and $\frac{c}{h} = 0.7$ for various values of V and $\frac{\ell}{R}$.

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